2. OSCILAŢII

2.1. Noţiuni generale

Se numeşte *oscilaţie* fenomenul fizic în decursul căruia o anumită mărime fizică a procesului prezintă o variaţie periodică sau pseudo-periodică în timp. Un sistem fizic izolat, care este pus în oscilaţie, efectuează *oscilaţii libere* sau proprii, cu o frecvenţă numită *frecvenţa proprie* a sistemului oscilant. Oscilaţiile pot fi clasificate în funcţie de mai multe criterii.

Din punct de vedere a formei de energie dezvoltată în timpul oscilaţiei, putem întâlni: (i) oscilaţii elastice, mecanice (au loc prin transformarea reciprocă a energiei cinetice în energie potenţială); (ii) oscilaţii electromagnetice (au loc prin transformarea reciprocă a energiei electrice în energie magnetică); (iii) oscilaţii electromecanice (au loc prin transformarea reciprocă a energiei mecanice în energie electromagnetică).

Din punct de vedere al conservării energiei sistemului oscilant, putem clasifica oscilaţiile în: (i) oscilaţii nedisipative, ideale sau neamortizate (energia totală se conservă); (ii) oscilaţii disipative sau amortizate (energia scade în timp); (iii) oscilaţii forţate sau întreţinute (se furnizează energie din afara sistemului, pentru compensarea pierderilor).

Mărimi caracteristice oscilaţiilor.

Să notăm cu S(t) mărimea fizică care caracteriozează o oscilaţie. Atunci, dacă T este perioada oscilaţiei, mărimea S are aceaşi valoare la momentul t şi la un moment ulterior, t + T:

$$S(t) = S(t+T)$$

.
.
.

Oscilaţiile armonice sunt acel tip de oscilaţii în care mărimile caracteristice se pot exprima prin funcţii trigonometrice (sinus, cosinus sau funcţii exponenţiale de argument complex). Acele oscilaţii care nu sunt armonice se pot descompune în serie Fourier de funcţii. Reamintim de asemenea formulele lui Euler, care vor fi utile în calculele următoare:

Mişcarea oscilatorie armonică apare foarte des în situaţiile practice. Un exemplu foarte la îndemână îl constituie bătăile inimii. Se spune că Galilei folosea bătăile inimii sale pentru a cronometra mişcările pe care le studia.

2.2. Mişcarea oscilatorie armonică ideală

În absenţa unor forţe de frecare sau de disipare a energiei, mişcarea oscilatorie este o mişcare ideală, deoarece energia totală a oscilatorului rămâne constantă în timp. Mişcarea este reversibilă, astfel că după o perioadă oscilatorul revine în poziţia iniţială şi procesul se reia. Forţa care determină revenirea oscilatorului în poziţia iniţială şi care permite continuarea oscilaţiei se numeşte *forţă de revenire*. Această forţă de revenire poate fi forţa elastică dint-o lamă metalică, presiunea dintr-un tub, etc.

Să considerăm un oscilator mecanic format dintr-un resort elastic şi un corp punctiform, de masă m, legat la capătul liber al resortului, ca în fig.2.1.a. Dacă se pune corpul în mişcare prin intermediul unei forţe şi dacă nu există frecări, sistemul va efectua o mişcare periodică în jurul poziţiei de echilibru, numită *oscilaţie ideală*.

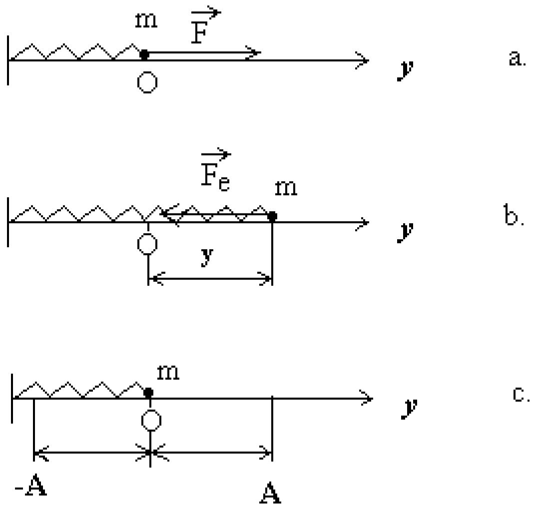
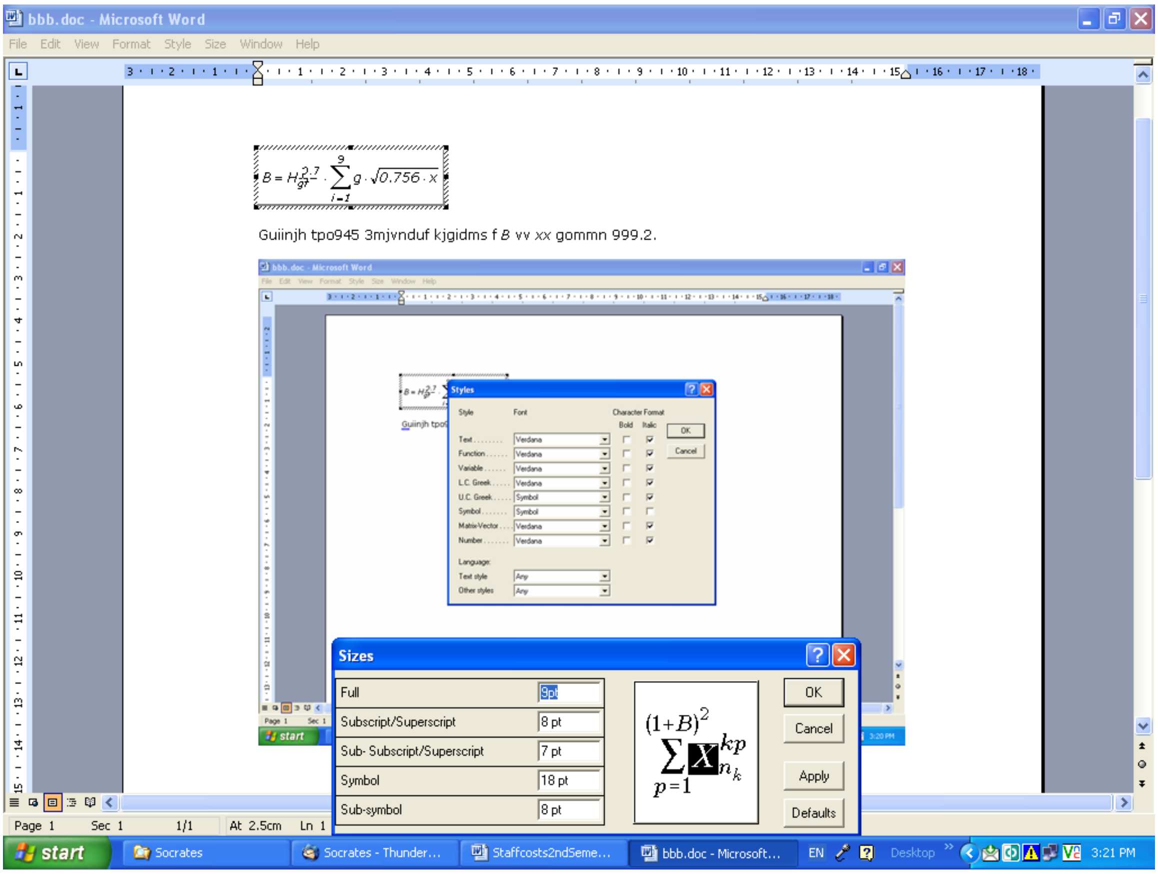


Fig.2.1. Oscilator mecanic ideal: a) momentul iniţial; b) alungirea y produce forţa de revenire $\vec{F}_e$; c) amplitudinea mişcării oscilatorii.

Detalii privind editarea formulelor:

$$B = H_{gf}^{0.7} \cdot \sum_{i=1}^{9} g \cdot \sqrt{0.756 \cdot x}$$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A Representation Theorem for the Error of Recursive Estimators

László Gerencsér

***Abstract*— The objective of this paper is to present advanced and less known techniques for the analysis of performance degradation due to statistical uncertainty for a wide class of linear stochastic systems in a rigorous and concise manner. The main technical advance of the present paper is a strong approximation theorem for the Djereveckii–Fradkov–Ljung (DFL) scheme with enforced boundedness, in which, for any $q \geq 1$, the L_q-norms of the so-called residual terms are shown to tend to zero with rate $N^{-1/2-\varepsilon}$ with some $\varepsilon > 0$. This is a significant extension of previous results for the recursive prediction error or RPE estimator of ARMA processes given in [L. Gerencsér, *Systems Control Lett.*, 21 (1993), pp. 347–351. Two useful corollaries will be presented. In the first a standard transform of the estimation-error process will be shown to be L-mixing. In the second the asymptotic covariance matrix of the estimator will be given. An application to the minimum-variance self-tuning regulator for ARMAX systems will be described.**

I. Introduction

The objective of this paper is to present new techniques for the analysis of performance degradation due to statistical uncertainty for a wide class of linear stochastic systems in a rigorous and concise manner. It is hoped that this paper helps to access the complete theory developed in [18].

Performance degradation due to statistical uncertainty is called *regret*, following [28]. The objective of the paper is to develop new techniques that can be used for analyzing the path-wise (almost sure) asymptotics of the cumulative regret for a class of adaptive prediction and stochastic adaptive control problems. Special examples of these technical tools have been presented in [16]. This research is also motivated by problems in *stochastic complexity* and *identification for control*, see [34] and [21].

The immediate technical objective is a detailed analysis of the Djereveckii–Fradkov–Ljung (DFL) scheme with enforced boundedness, given as *Algorithm DFL*, (21)–(22); see [7], [33]. This is a practically useful recursive estimation method with a wide range of applications; see [7], [33].

The study of the DFL scheme can be reduced to the study of two related stochastic approximation methods, Algorithm DR (discrete-time recursion) and Algorithm CR (continuous-time recursion). Therefore some of the results will be stated only for Algorithm CR.

Tight control of the difference between the estimation error and its standard approximation, that will be referred to as residuals, is crucial in the analysis of performance degradation due to statistical uncertainty; see [16]. The *main technical advance* of the present paper is a strong approximation theorem for the DFL scheme, given as Theorem 4.2. It extends the result of [15] on the residual of the recursive prediction error estimator for ARMA processes.

The proof relies on [12] and uses a nontrivial moment inequality for weighted multiple integrals of L-mixing processes given in [14]. Preliminary versions of the results of have been formulated in [13]. In Theorem 5.1 a standard transform of the estimation-error process for the basic recursive estimation method, Algorithm CR, is shown to be an L-mixing process, while in Theorem 6.1 the asymptotic covariance matrix of the estimator for the same method will be given.

The *significance* of the results of the present paper is demonstrated by describing an applications in Section 7, in which the path-wise cumulative regret for the minimum-variance self-tuning regulator is computed.

II. Basic notions and conditions

We shall need the following definition, see [10]. We say that a discrete-time $\mathbb{R}^p$-valued stochastic process (u_n) is M-bounded if, for all $1 \leq q < \infty$,

$$M_q(u) := \sup_{n \geq 0} \mathrm{E}^{1/q} |u_n|^q < \infty. \quad (1)$$

In this case we also write $u_n = O_M(1)$. For a stochastic process $(z_n), n \geq 0$, and a positive sequence (c_n) we write $z_n = O_M(c_n)$ if $u_n = z_n/c_n = O_M(1)$.

A basic tool that we will use is the theory of L-mixing processes, see [10], that has been successfully applied in [11], [12], [15], [23], [26]. For a similar notion see Definition 3.1 in Section 8.3 of [3]. Let a probability space $(\Omega, \mathcal{F}, P)$ be given together with a pair of families of σ-algebras $(\mathcal{F}_n, \mathcal{F}_n^+), n = 0, 1, \ldots,$ such that (i) $\mathcal{F}_n \subset \mathcal{F}$ is monotone increasing, (ii) $\mathcal{F}_n^+ \subset \mathcal{F}$ is monotone decreasing, and (iii) $\mathcal{F}_n$ and $\mathcal{F}_n^+$ are independent for all n. For $n < 0$ we set $\mathcal{F}_n^+ = \mathcal{F}_0^+$.

Definition 2.1: A stochastic process $u = (u_n), n \geq 0$, is L-mixing with respect to $(\mathcal{F}_n, \mathcal{F}_n^+)$ if it is $\mathcal{F}_n$-adapted, M-bounded, and for all $q \geq 1$, with $\tau \geq 0$ and

$$\gamma_q(\tau, u) = \gamma_q(\tau) = \sup_{n \geq \tau} \mathrm{E}^{1/q} |u_n - \mathrm{E}\,(u_n | \mathcal{F}_{n-\tau}^+)|^q,$$

we have

$$\Gamma_q = \Gamma_q(u) = \sum_{\tau=0}^{\infty} \gamma_q(\tau) < \infty. \quad (2)$$

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MTA SZTAKI (Computer and Automation Research Institute of the Hungarian Academy of Sciences), Kende 13-17, H-1111 Budapest, Hungary (gerencser@sztaki.hu).

The process u is L^+-mixing if, in addition, for all $q \geq 1$ there exist $C_q, c_q > 0$ such that for all $\tau \geq 0$,

$$\gamma_q(\tau, u) \leq C_q(1+\tau)^{-1-c_q}.$$

The *verification* of L-mixing is typically easy in problems of system identification, in contrast to other notions of mixing, such as ϕ-mixing, see, e.g., Chapter 7.2 of [9]. For this see see Chapter 17 of [25]. The prime example for L-mixing processes is a sequence of i.i.d. random variables with finite moments of all orders. The response of an exponentially stable linear filter, with an L-mixing process as its input, is L-mixing. Products of L-mixing processes are also L-mixing.

III. General recursive estimation schemes

The prime objective of this section is to formulate a general recursive estimation method, the DFL scheme with *enforced boundedness*, together with conditions that ensure its convergence. But first we present two closely related recursive algorithms which can be interpreted as "frozen parameter" approximations to the DFL scheme. The connection between Algorithm DR and the DFL scheme is not straightforward at all, and will be discussed in some detail.

Our continuous-time recursive estimation process is given by a random differential equation of the form

$$\dot{x}_t = \frac{1}{t}(H(t, x_t, \omega) + \delta H(t, \omega)), \qquad x_1 = \xi_1, \qquad (3)$$

defined over an underlying probability space $(\Omega, \mathcal{F}, P)$. Here x_t is the estimator sequence and $H = (H(t, x, \omega))$ is a random field defined in $[1, \infty) \times D \times \Omega$, where D is a bounded open domain in $\mathbb{R}^p$ and $\delta H(t, \omega))$ is a perturbation term. The technical conditions are identical with those of [18].

Condition 3.1: The process $H = (H(t, x, \omega))$ is defined in $\Omega \times \mathbb{R}^+ \times D$, where $D \subset \mathbb{R}^p$ is an open domain. It is three times continuously differentiable with respect to x for $x \in D$ and for all ω, and for any compact set $D_0 \subset D$ H and its derivatives up to order 3 are M-bounded in D_0. Furthermore $(H(t, x, \omega))$ and its first derivative $H_x = (H_x(t, x, \omega))$ are L^+-mixing with respect to $(\mathcal{F}_t, \mathcal{F}_t^+)$, uniformly in $x \in D_0$.

Condition 3.2: $H(t, x, \omega)$ is piecewise continuous in t for all ω, and for any compact set $D_0 \subset D$ there exists a random variable $L_t = L_t(\omega) \geq 0$ such that

$$|H_x(t, x, \omega)| \leq L_t(\omega)$$

for all $x \in D_0$, and here L_t is such that for some $\varepsilon > 0$ we have

$$\sup_t \mathrm{E} \exp(\varepsilon L_t) < \infty. \qquad (4)$$

It follows that if $(\delta H(t, \omega))$ is piecewise continuous in t for all ω, then a solution (x_t) of (3) exists for all ω in some finite or infinite interval. A central role in the analysis of (x_t) is played by the mean-field $\mathrm{E}H(t, x, \omega)$.

Condition 3.3: We have for any compact set $D_0 \subset D$ and $t \geq 0$, $x \in D_0$

$$\mathrm{E}H(t, x, \omega) = G(x) + \delta G(t, x),$$

where $\delta G(t, x) = O(t^{-1/2-\varepsilon})$ uniformly in $x \in D_0$, with some $\varepsilon > 0$. $G(y)$ has continuous and bounded partial derivatives up to third order. Finally, we assume that

$$G(x) = 0 \qquad (5)$$

has a unique solution x^* in D.

The celebrated "ODE principle" states that the solution trajectories of the random differential equation (3), under additional conditions, follow the solution trajectories (x_t) of the associated ODE (6) given by

$$\dot{y}_t = \frac{1}{t} G(y_t), \qquad y_s = \xi, \; s \geq 1. \qquad (6)$$

Under the conditions above, (6) has a unique solution in some finite or infinite interval, which we denote by $y(t, s, \xi)$. Since H is not defined on the whole space, we have to make sure that the process (x_t) is constrained to D by a *resetting* mechanism.

Algorithm CR. Consider a continuous-time recursion given by a random differential equation

$$\dot{x}_t = \frac{1}{t}(H(t, x_t, \omega) + \delta H(t, \omega)), \qquad x_1 = \xi_1 \qquad (7)$$

combined with the following *resetting* mechanism. Let $D_0 \subset D$ denote a compact truncation domain such that $x^* \in$ int D_0. Let $\sigma \geq 1$ and let

$$\tau(\sigma) = \min\{t : t > \sigma, x_t \in \partial D_0\}, \qquad (8)$$

where ∂D_0 denotes the boundary of D_0. Then we reset x to $x_1 = \xi_1$, which is formally stated by requiring that the right-hand side limit of x_t at $t = \tau = \tau(\sigma)$ will be ξ_1:

$$x_{\tau+} = \xi_1. \qquad (9)$$

Thus we get a piecewise continuous trajectory (x_t) defined in some finite or infinite interval.

To ensure that the estimator sequence is not bounced back and forth by resetting we need to impose some condition on relative position of x^* and ξ_1 to the truncation domain. We define the star-like closure of the set D_0, relative to x^* as

$$D_0^* = \{y : y = x^* + \lambda(x - x^*), \; 0 \leq \lambda \leq 1, x \in D_0\}.$$

Condition 3.4: Let $D_0 \subset D$ be a compact truncation domain such that $x^* \in \mathrm{int} D_0$. We assume that (i) D is convex and there exists a compact set $D_0' \subset D$ such that $y(t, s, \xi) \in D_0'$ for $\xi \in D_0$ and $y(t, s, \xi) \in D$ for $\xi \in D_0'$ for all $t \geq s \geq 1$. In addition $\lim_{t \to \infty} y(t, s, \xi) = x^*$ for $\xi \in D$ and

$$\|(\partial/\partial\xi) y(t, s, \xi)\| \leq C_0 (s/t)^\alpha \qquad (10)$$

with some $C_0 \geq 1, \alpha > 0$ for all $\xi \in D_0'$ and $t \geq s \geq 1$. (ii) We have an initial estimate $x_1 = \xi_1$ such that for all $t \geq s \geq 1$ we have $y(t, s, \xi_1) \in$ int D_0. (iii) Finally, for the star-like closure of the set D_0 we have $D_0^* \subset D$.

On (10): it can be shown that with

$$A^* = \frac{\partial G(x)}{\partial x}\Big|_{x=x^*} \qquad (11)$$

and

$$\alpha^* = \min_i\{-\Re\lambda_i(A^*)\}, \quad (12)$$

where $\lambda_i(A^*)$ are the eigenvalues of A^*, condition (10) holds with

$$\alpha = \alpha^*_-, \quad (13)$$

where α^*_- denotes any number that is smaller than α^*. We will also use the notation

$$\overline{\alpha} = \alpha - 1/2. \quad (14)$$

Finally, consider the perturbation term $\delta H(t,\omega)$. Following [12] we use the following condition that will be discussed later.

Condition 3.5: $(\delta H(t,\omega))$ is a measurable M-bounded process, which is piecewise continuous in t for all ω, moreover there exists an $\varepsilon > 0$ such that for any fixed $q > 1$ and for any $s \geq 1$,

$$\sup_{s\leq\sigma\leq qs} \int_\sigma^{\tau(\sigma)\wedge q\sigma} \frac{1}{r}|\delta H(r,\omega)|dr = O_M(s^{-1/2-\varepsilon}). \quad (15)$$

It is no loss of generality to assume that $\varepsilon < 1/2$. We assume that the ε's showing up here and in Condition 3.3 are identical.

Then a discrete-time recursive estimation process is given by the following algorithm:

Algorithm DR. (Discrete-time recursion with resetting):

$$x_{n+1} = x_n + \frac{1}{n+1}(H(n+1,x_n,\omega) + \delta H(n+1,\omega)), \quad (16)$$

with $x_0 = \xi_0 \in \text{ int } D_0$. Boundedness of the estimator sequence is enforced by resetting x_{n+1} to x_0 if the pre-computed value of x_{n+1} leaves D_0.

We can analyze this algorithm by continuous-time imbedding, see [12]. A more accurate and more recent technique is to use a discrete-time ODE method, see [17].

We now turn to the definition of the DFL scheme, based on [5], [6], [32], see also the books [2], [7], [33]. Its basic building block is a parameter-dependent $\mathbb{R}^r$-valued process $(\overline{\phi}_n(x))$, with $x \in D \subset \mathbb{R}^p$ defined by the state-space equation

$$\overline{\phi}_{n+1}(x) = A(x)\overline{\phi}_n(x) + B(x)e_n, \quad (17)$$

with some non-random initial condition $\overline{\phi}_0(x)$, the value of which is often assumed to be zero. To ensure that for any choice of $x = x_n \in D$ the time-varying system obtained from (17) is bounded input-bounded output stable, we need the following condition.

Condition 3.6: The family of matrices $A(x)$, $x \in D_0$, with D_0 being the preselected truncation domain, is jointly stable in the sense that there exist a single symmetric positive definite $r \times r$ matrix V and $0 < \lambda < 1$ such that

$$A^T(x)VA(x) \leq \lambda V \quad \text{for all } x \in D_0.$$

Moreover, the functions $A(x), B(x)$ are three times continuously differentiable in D.

Joint stability can be achieved by suitable x-dependent state-space transformations that convert all A-matrices into contractions.

Condition 3.7: We assume that (e_n) is a wide-sense stationary process and that $|e_n|^2$ is such that for some $\varepsilon > 0$ we have

$$\sup_n \mathrm{E}\ \exp \varepsilon|e_n|^2 < \infty.$$

Condition 3.8: We assume that (e_n) is L^+-mixing with respect to a pair of families of σ-algebras $(\mathcal{F}_n, \mathcal{F}_n^+)$.

Discussion. Condition 3.7 is standard in [4]. The weaker condition that (e_n) is M-bounded is implicitly assumed also in [2] (see Example 1, p. 215). Condition 3.7 is essential to ensure that the application of the Bellman–Gronwall lemma to estimate $x_t - y_t$ gives meaningful result. The role of Condition 3.8 is essential in establishing the connection between the DFL scheme and the "frozen parameter" algorithm Algorithm DR, see [12, sections 5 and 6].

Define a random field $H(n,x,\omega)$ as follows:

$$H(n,x,\omega) = Q(\overline{\phi}_n(x)), \quad (18)$$

where for the sake of simplicity Q is a quadratic function from $\mathbb{R}^r$ to $\mathbb{R}^p$. An alternative, more general definition would be $H(n,x,\omega) = F(Q(\overline{\phi}_n(x)),x)$, where Q is quadratic and F is three times continuously differentiable and linear in Q. We define the mean-field $G(x)$ as above. The estimation problem in the context of the DFL scheme is then to solve the nonlinear algebraic equation

$$G(x) = 0$$

in real time. It is assumed that a unique solution x^* exists in D.

The estimate of x^* at time n will be denoted by x_n. We generate an online approximation of $Q(\overline{\phi}_n(x_n))$ and thus we arrive at the following tentative first version of the DFL method:

$$\phi_{n+1} = A(x_n)\phi_n + B(x_n)e_n, \quad (19)$$

$$x_{n+1} = x_n + \frac{1}{n}Q(\phi_{n+1}), \quad (20)$$

with initial conditions $x_0 = \xi_0 \in \text{int}D_0$ and ϕ_0 a constant, nonrandom initial state. It is assumed that $Q(\phi_{n+1})$ is *computable* by coupling a physical system with our computer.

Discussion on the DFL scheme. The applicability of this general estimation scheme in the theory of recursive identification of linear stochastic systems has been discussed in more details [33]. Further examples of application are given in [2], in which also a rigorous and detailed analysis of a nonlinear modification of the DFL scheme is given, using a Markovian dynamics in generating the state sequence (ϕ_n). A special example of the extended DFL scheme is the LMS method, in which $\overline{\phi}_n$ does not depend on x at all.

It is well known from simulations that the DFL scheme may diverge, unless some precaution is taken. Therefore the estimates x_n will be enforced to stay in a compact domain $D_0 \subset D$, such that $x^* \in \text{int}D_0$. This will be achieved by the

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usual a *resetting* mechanism. Thus we arrive at the following algorithm:

Algorithm DFL. (The DFL scheme with resetting):

$$\phi_{n+1} = A(x_n)\phi_n + B(x_n)e_n, \quad (21)$$

$$x_{n+1-} = x_n + \frac{1}{n+1}Q(\phi_{n+1}), \quad (22)$$

and if $x_{n+1-} \notin \mathrm{int} D_0$, then we reset it to ξ_0.

Discussion of the "boundedness condition." The eventual divergence of the DFL scheme is traditionally dealt with a controversial boundedness condition, or an equally controversial "projection method" which may fail even for deterministic algorithms. A rigorous treatment of the boundedness problem has been given in [2], where the estimator process is stopped if it leaves a prescribed compact domain. Denoting by $\Omega' \subset \Omega$ the event that the estimator process is never stopped, the almost sure convergence of the estimator process has been established on Ω'; see [2, Part II, Chapter 1.6, Proposition 11].

Discussion of Condition 3.5. To connect the DFL scheme with Algorithm DR define

$$\delta H(n, \omega) = Q(\phi_n) - Q(\overline{\phi}_n(x_n)). \quad (23)$$

Then (22) can be written in the form of (16). A critical point in the analysis of the DFL scheme is that the perturbation term $\delta H(n, \omega)$ is *not given a priori*, rather it is defined via the recursive procedure itself. The analysis of $\delta H(n, \omega)$ is a substantial component of the convergence analysis of the DFL-method, which has been worked out in [12, sections 5 and 6], leading to the following result:

Consider the DFL scheme defined by (21)–(22). Assume that Conditions 3.6, 3.7, and 3.8 are satisfied. In addition assume that Condition 3.4 is satisfied with $\alpha > 1/2$. Then $(\delta H(n, \omega))$ defined by (23) is an M-bounded process, moreover $(\delta H(n, \omega))$ satisfies the discrete version of Condition 3.5.

A basic result if [12] is the following theorem, a special feature of which is that the *higher order moments* of the estimation error are bounded from above.

Theorem 3.1: Consider the DFL scheme defined by (21)–(22). Assume that Conditions 3.6, 3.7, and 3.8 are satisfied. In addition assume that Condition 3.4 is satisfied with $\alpha > 1/2$. Then we have $x_n = O_M(n^{-1/2})$.

IV. Strong Approximation of the Estimation Error

In this section we present a significant extension of the results of [15] for the recursive estimation schemes of the previous section.

Theorem 4.1: Consider the continuous-time recursive estimation scheme, Algorithm CR, given by (7) with the resetting mechanism (8) and (9). Assume that Conditions 3.1–3.5 are satisfied and Condition 3.4 is satisfied with $\alpha > 1/2$. Then the solution of (7), (x_t), is defined for all $t \in [1, \infty)$ with probability 1 and we have with

$$\varepsilon_x = \min(\overline{\alpha}, \varepsilon)_-,$$

where c_- is any number smaller than c, $\overline{\alpha}$ is given by (14), and ε is given in Condition 3.5,

$$x_t - x^* = \int_1^t \frac{\partial}{\partial \xi} y(t, s, x^*) \frac{1}{s} H(s, x^*, \omega) ds + O_M(t^{-1/2-\varepsilon_x}).$$

To interpret this result note that the matrix $(\frac{\partial}{\partial \xi}) y(t, s, x^*)$ is the sensitivity matrix, which indicates the relative effect of a perturbation of the initial condition at time s on the solution of (6) at time t. Thus the dominant term on the right-hand side represents the cumulative effect of the ideal correction terms $\frac{1}{s}H(s, x^*, \omega)$ at time t. A relatively straightforward corollary of Theorem 4.1 is an analogous discrete-time result, which, specialized to the DFL scheme, gives the following:

Theorem 4.2: Consider the DFL scheme defined by (21)–(22). Assume that the state-space equation (17) satisfies Condition 3.6, the noise process (e_n) satisfies Conditions 3.7 and 3.8, and the associated ODE satisfies Condition 3.4 with $\alpha > 1/2$. Let $\varepsilon_x = \min(\overline{\alpha}, \varepsilon)_-$, where $\overline{\alpha}$ is defined under (14) and ε is given in Condition (3.5). Then we have

$$x_N - x^* = \sum_{n=1}^{N} \frac{\partial y}{\partial \xi}(N, n, x^*) \frac{1}{n} Q(\overline{\phi}_n(x^*)) + O_M(N^{-1/2-\varepsilon_x}).$$

The above result is particularly simplified for partially *stochastic Newton methods*, i.e. when the Jacobian A^* is of the form

$$\begin{pmatrix} -I & 0 \\ X & Y \end{pmatrix},$$

where I is an identity matrix. The corresponding decomposition of the parameter vector x is $x = (x^1, x^2)$.

Theorem 4.3: Assume that the conditions of Theorem 4.2 are satisfied and that we can split the parameter vector x as $x = (x^1, x^2)$ so that the estimation method is a partially stochastic Newton method with respect to x^1. Let (Q^1, Q^2) be the corresponding splitting of Q. Then

$$x_N^1 - x^{1*} = \frac{1}{N} \sum_{n=1}^{N} Q^1(\overline{\phi}_n(x^*)) + O_M(N^{-1/2-\varepsilon_x}).$$

Discussion of the result. Theorem 4.3 is a powerful alternative to results obtained by weak convergence techniques, see [27], or by stochastic regression methods, see [29], [8].

The proof of the key Theorem 4.1 relies on a moment inequality for weighted multiple integrals of L-mixing processes given in [14].

V. The Transformed Error Process is L-Mixing

In this section present an extension of one of the results in [16] stating that an appropriate transformation of the error process $x_t - x^*$ is L-mixing. Define the transformed process

$$\tilde{x}_r = e^{r/2}(x_{e^r} - x^*). \quad (24)$$

The weak limit of the shifted process $(\tilde{x}_{r+\rho})$, when $\rho \to \infty$, is studied in Chapter 4.5, Part II of [2](see Theorem 13) in a Markovian framework. It is proven there that $(\tilde{x}_{r+\rho})$ converges weakly to the stationary solution of the linear stochastic differential equation

$$d\tilde{z}_r = (A^* + I/2)\tilde{z}_r + d\tilde{w}_r, \quad (25)$$

assuming that $(A^* + I/2)$ is stable. Here $d\tilde{w}_r$ is a Gaussian white noise with intensity, say $P^* dt$. - The weak limit $(\tilde{z}_r)$ is a prime example for an L-mixing process. A surprising result is that the transformed process $(\tilde{x}_r)$ itself is also L-mixing.

Theorem 5.1: Consider the continuous-time recursive estimation scheme given by (7) with the resetting mechanism (8) and (9). Assume that the conditions of Theorem 4.1 are satisfied. Then the transformed process $(\tilde{x}_r)$ is L-mixing with respect to $(\mathcal{F}_{e^r}, \mathcal{F}^+_{e^r})$.

VI. THE ASYMPTOTIC COVARIANCE MATRIX

The asymptotic covariance matrix for Algorithm DFL, has been rigorously derived in Chapter 4.5, Part II of [2] in a *series* model, where the initial time tends to infinity. The main advance of this section relative to the cited result is that the asymptotic covariance matrix is obtained for a *single* process, defined by Algorithm CR, with resetting. We need the following condition:

Condition 6.1: We assume that $(H(s, x^*, \omega))$ is asymptotically wide-sense stationary in the following sense: there exists a zero-mean, wide-sense stationary process $(H_0(s, x^*, \omega))$ such that with some $\varepsilon_H > 0$.

$$\eta_s = H(s, x^*, \omega) - H_0(s, x^*, \omega) = O_M(s^{-1-\varepsilon_H}). \quad (26)$$

There is no loss of generality to assume that

$$\gamma_q(\tau, H_0) \leq C_q(1+\tau)^{-1-c_q}$$

for all $\tau \geq 0$ with the same C_q, c_q as for H in Condition 3.1.

Denoting the auto-covariance matrix of $H_0(s, x^*, \omega)$ by $\rho(\tau)$, i.e., setting

$$\rho(\tau) = \mathrm{E}\left[H_0(s+\tau, x^*, \omega) H_0^T(s, x^*, \omega)\right],$$

we define a basic quantity:

$$P^* = \int_{-\infty}^{\infty} \rho(\tau) d\tau. \quad (27)$$

It is easy to see that the matrix P^* is the asymptotic covariance matrix of the arithmetic means

$$\frac{1}{2T} \int_{-T}^{T} H_0(s, x^*, \omega) ds.$$

The value of the asymptotic covariance matrix can be easily guessed, assuming the validity of $(\tilde{x}_{r+\rho}) \rightarrow (\tilde{z}_r)$ (in a weak sense) under our set of conditions.

Theorem 6.1: Consider the continuous-time recursive estimation scheme given by (7) with the resetting mechanism (8) and (9). Assume that the conditions of Theorem 4.1 are satisfied and in addition $(H(s, x^*, \omega))$ satisfies Condition 6.1. Let A^* be as defined in (12) Then the asymptotic covariance matrix of the error process $(x_t - x^*)$, defined by

$$S^* = \lim_{t \to \infty} t \mathrm{E}[(x_t - x^*)(x_t - x^*)^T],$$

exists and it satisfies the Lyapunov equation

$$(A^* + I/2) S^* + S^* (A^* + I/2)^T + P^* = 0. \quad (28)$$

Also we have with some $\varepsilon_{xx} > 0$

$$\mathrm{E}[(x_t - x^*)(x_t - x^*)^T] = \frac{1}{t} S^* + O(t^{-1-\varepsilon_{xx}}).$$

For a stochastic Newton method, i.e. when $A^* = -I$, we get

$$S^* = P^*.$$

VII. AN APPLICATION

The minimum-variance self-tuning regulator. The performance degradation of the minimum-variance self-tuning regulator were first rigorously studied in [30], [31] and [22]. In these papers the right order of magnitude for the so-called cumulative regret was found for ARMAX systems, and in [31] the asymptotic value of the normalized regret was also found for ARX systems. We extend the latter result to ARMAX systems.

Consider a stochastic control system in ARMAX(n, m, p) representation defined by the relation

$$A^*(q^{-1}) y = q^{-1} B^*(q^{-1}) u + C^*(q^{-1}) e, \quad (29)$$

where $A^*(q^{-1})$, $B^*(q^{-1})$, and $C^*(q^{-1})$ are polynomials of the backward shift operator q^{-1} of degree n, m, p, respectively. The minimum-variance control is given by (cf. [1])

$$u(t-1) = -(\eta^*)^T \phi(t), \quad (30)$$

with

$$\phi(t) = (-y(t-1), \ldots, -y(t-n), u(t-2), \ldots, u(t-m-1)),$$

and the optimal controller is

$$\eta^* = \frac{1}{b_0^*}(a_1^* - c_1^*, \ldots, a_n^* - c_n^*, b_1^*, \ldots, b_m^*)^T.$$

Let $D \subset \mathbb{R}^{n+m}$ be a set of feasible controller parameters to be specified below. For *any* $\eta \in D$ and for $t \geq 0$ we consider the control law

$$u(t-1) = -\eta^T \phi(t),$$

where $\phi(t)$ is defined above. Thus we get a closed-loop system with inputs and outputs

$$u(t) = \bar{u}(t, \eta) \quad \text{and} \quad y(t) = \bar{y}(t, \eta).$$

Let D denote the open set of η's in $\mathbb{R}^{m+n}$ such that the closed-loop system is stable. Define

$$G(\eta) \triangleq \lim_{t \to \infty} \mathrm{E}\left[\bar{\phi}(t, \eta) \bar{y}(t, \eta)\right]. \quad (31)$$

For $\eta = \eta^*$ we have, modulo negligible errors, $\bar{y}(t, \eta) = e(t)$, and thus $G(\eta^*) = 0$.

Let $\hat{\eta}(t)$ be the recursive estimator of η^* defined by the self-tuning regulator, defined in [1], and modified by a resetting mechanism, and let S^* denote the asymptotic covariance matrix of $\hat{\eta}(t)$; i.e., let

$$S^* = \lim_{t \to \infty} t \cdot \mathrm{E}\left[(\hat{\eta}(t) - \eta^*)(\hat{\eta}(t) - \eta^*)^T\right].$$

The existence of S^* is guaranteed by Theorem 6.1. Define the *second order sensitivity matrix*

$$T^* = \lim_{t \to \infty} \mathrm{E}\left[\frac{\partial^2}{\partial \eta^2}_{|\eta=\eta^*} \bar{y}^2(t, \eta)\right]. \quad (32)$$

CLAIM. *Under appropriate technical conditions, obtained by specializing the conditions of Theorem 4.2, we have*

$$\lim_{N\to\infty}\sum_{t=1}^{N}(y^2(t)-e^2(t))/\log N=\frac{1}{2}\text{Tr}\,T^*S^* \qquad (33)$$

almost surely. Moreover, we have

$$\frac{1}{2}\text{Tr}\,T^*S^*\geq\sigma^2(e)(m+n). \qquad (34)$$

The inequality (34) *is an equality if and only if* $C^*=1$ *and the updating of* $\hat{\eta}(t)$ *is done by a stochastic Newton-method.*

Remark. For a stochastic Newton-method we would need the matrix $R^*=-G_\eta(\eta^*)$, which had been believed to be uncomputable prior to the work of Hjalmarsson; cf. [24]. However, in [24] it has been shown that for certain interesting physical systems $G_\eta(\eta^*)$ is in fact computable.

VIII. CONCLUSION

Performance degradation due to statistical uncertainty, also called regret, is of great interest in adaptive prediction and control of stochastic systems. To quantify the pathwise cumulative regret we need technical tools similar to those developed in [16] in the context of adaptive prediction of ARMA processes. These new tools have been developed in this paper. The usefulness of the results in stochastic adaptive control has been demonstrated for the minimum-variance self-tuning regulator for ARMAX systems. A further application for indirect adaptive control of multivariable linear stochastic systems is given in [20].

The results can be also applied in the context of *identification for control*; see [21]. A further potential area of application is adaptive experimental design, see [19]. The extension of the results of the present paper to Kiefer–Wolfwitz-type procedures, such as the simultaneous perturbation stochastic approximation, or SPSA, method due to Spall [35] seems to be possible.

REFERENCES

[1] K.J. Åström and B. Wittenmark. Problems of identification and control. *J. Math. Anal. Appl.*, 34:90–113, 1971.

[2] A. Benveniste, M. Métivier, and P. Priouret. *Adaptive algorithms and stochastic approximations*. Springer-Verlag, Berlin, 1990.

[3] P.E. Caines. *Linear Stochastic Systems*. Wiley, 1988.

[4] H.F. Chen and L.Guo. *Identification and Stochastic Adaptive Control*. Birkhauser, 1991.

[5] D.P. Djereveckii and A.L. Fradkov. Application of the theory of Markov-processes to the analysis of the dynamics of adaptation algorithms. *Automation and Remote Control*, (2):39–48, 1974.

[6] D.P. Djereveckii and A.L. Fradkov. Two models for analysing the dynamics of adaptation technics. *Automation and Remote Control*, (1):67–75, 1974.

[7] D.P. Djereveckii and A.L. Fradkov. *Applied theory of discrete adaptive control systems*. Nauka, Moscow, 1981. In Russian.

[8] T.E. Duncan and B. Pasik-Duncan. Some methods for the adaptive control of continuous time linear stochastic systems. In L. Gerencsér and P.E. Caines, editors, *Topics in Stochastic Systems: Modelling, Estimation and Adaptive Control*, pages 242–267. Springer-Verlag Berlin, Heidelberg, 1991.

[9] S.N. Ethier and T.G.Kurtz. *Markov Processes. Characterization and Convergence*. Wiley, 1986.

[10] L. Gerencsér. On a class of mixing processes. *Stochastics*, 26:165–191, 1989.

[11] L. Gerencsér. On the martingale approximation of the estimation error of ARMA parameters. *Systems & Control Letters*, 15:417–423, 1990.

[12] L. Gerencsér. Rate of convergence of recursive estimators. *SIAM J. Control and Optimization*, 30(5):1200–1227, 1992.

[13] L. Gerencsér. A representation theorem for the error of recursive estimators. In *Proc. of the 31st IEEE Conference on Decision and Control, Tucson*, pages 2251–2256, 1992.

[14] L. Gerencsér. Multiple integrals with respect to *L*-mixing processes. *Statistics and Probability Letters*, 17:73–83, 1993.

[15] L. Gerencsér. Strong approximation of the recursive prediction error estimator of the parameters of an ARMA process. *Systems & Control Letters*, 21:347–351, 1993.

[16] L. Gerencsér. On Rissanen's predictive stochastic complexity for stationary ARMA processes. *Statistical Planning and Inference*, 41:303–325, 1994.

[17] L. Gerencsér. Stability of random iterative mappings. In M. Dror, P. L'ecuyer, and F. Szidarovszky, editors, *Modeling Uncertainty. An Examination of its Theory, Methods, and Applications*, pages 359–371. Dordrecht, Kluwer, 2002.

[18] L. Gerencsér. A representation theorem for the error of recursive estimators. *Submitted to SIAM J. Control and Optimization*, 44:2123–2188, 2005.

[19] L. Gerencsér and H. Hjalamrsson. Adaptive input design in system identification. In *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference, CDC-ECC'05, Seville, Spain, December 12-15, 2005*, pages 4988–4993. IEEE, 2005.

[20] L. Gerencsér and Zs. Vágó. Adaptive control of multivariable linear stochastic systems. a strong approximation approach. In *Proceedings of the European Control Conference, ECC'99, Karlsuhe, Germany*, page F587, 1999.

[21] M. Gevers. Towards a joint design of identification and control ? In H.L. Trentelman and J.C. Willems, editors, *Essays on Control : Perspectives in the Theory and its Applications*, pages 111–151. Birkhäuser, 1993.

[22] L. Guo. Further results on least squares based adaptive minimum variance control. *SIAM J. on Control and Optimization*, 32(1):187–212, 1994.

[23] A. Heunis. Rates of convergence for an adaptive filtering algorithm driven by stationary dependent data. *SIAM J. on Control and Optimization*, 32:116–139, 1994.

[24] H. Hjalmarsson. Efficient tuning of linear multivariable controllers using iterative feedback tuning. *Int. J. Adapt.Control Signal Process.*, 13:553–572, 1999.

[25] I.A. Ibragimov and Yu. A. Linnik. *Independent and stationary sequences of random variables*. Wolters and Nordhoff, Groningen, 1971.

[26] J. A. Joslin and A. J. Heunis. Law of the iterated logarithm for a constant-gain linear stochastic gradient algorithm. *SIAM J. on Control and Optimization*, 39:533–570, 2000.

[27] H.J. Kushner and G. Yin. *Stochastic Approximation Algorithms and Applications*. Springer Verlag. New York, 1997.

[28] T.Z. Lai. Information bounds, certainty equivalence and learning in asymptotically efficient adaptive control of time-invariant stochastic systems. In L. Gerencsér and P.E. Caines, editors, *Topics in Stochastic Systems: Modelling, Estimation and Adaptive Control*, pages 268–299. Springer-Verlag Berlin, Heidelberg, 1991.

[29] T.Z. Lai and C.Z. Wei. Least squares estimates in stochastic regression models with application to identification and control of dynamic systems. *Ann. of Stat.*, 10:154 – 165, 1982.

[30] T.Z. Lai and C.Z. Wei. Extended Least Squares and their applications to adaptive control of dynamic systems. *IEEE Trans. Automat. Contr.*, 31:898–906, 1986.

[31] T.Z. Lai and C.Z. Wei. Asymptotically efficient self-tuning regulators. *SIAM J. Control and Optimization*, 25:466–481, 1987.

[32] L. Ljung. Analysis of recursive stochastic algorithms. *IEEE Trans. Automat. Contr.*, 22:551–575, 1977.

[33] L. Ljung and T. Söderström. *Theory and practice of recursive identification*. The MIT Press, 1983.

[34] J. Rissanen. *Stochastic complexity in statistical inquiry*. World Scientific Publisher, 1989.

[35] J.C. Spall. Multivariate stochastic approximation using a simultaneous perturbation gradient approximation. *IEEE Trans. Automat. Contr.*, 37:332–341, 1992.