PROBLEMS OF MATHEMATICS WITH SOLUTIONS

MATEMATICS - IX
(symbol AL – IX)

AL - IX. 001 Find the real numbers *a* with property

$\left[a+\frac{1}{2}\right]=\frac{5a-1}{3}$, and the interval which contains the solution.

a) $\left[\frac{3}{5},1\right]$ *b)* $\left[\frac{1}{5},\frac{4}{5}\right]$ *c)* $\left(\frac{1}{5},\frac{4}{5}\right)$

d) $\left[\frac{1}{5},\frac{3}{5}\right)$ *e)* $\left[0,\frac{2}{5}\right]$ *f)* $[1,\infty)$

Solution:7

We designate $\frac{5a-1}{3}=K$, hence $K\in N$. We have 5a – 1 = 3K, $a=\frac{3K+1}{5}$,

and

Hence $\left[\frac{6K+7}{10}\right]=K$. But $K\le\frac{6K+7}{10}<K+1$ and we obtain $a\in\left\{\frac{1}{5},\frac{4}{5}\right\}$.

Correct answer b.

AL - IX. 004 Solve the equation

$$5\left[x^2\right]-3[x]+2=0$$

a) $x\in\left[1,\sqrt{2}\right)$ b) $x\in\left(1,\sqrt{2}\right)$ c) $x\in(0,1)$

d) $x\in(0,1]$ e) $x\in\phi$ f) $x\in\left[\sqrt{2},2\right)$

Solution: We have:

(1) $x-1<[x]\le x,\ \forall\ x\in N$

(2) $x^2-1<\left[x^2\right]\le x^2,\ \forall\ x\in N$.

We multiplicate relation (1) by -3 and relation (2) by 5 and it results

(3) $-3x \le -3[x] < -3x+2$

(4) $5x^2 - 5 < 5[x^2] \le 5x^2$; If we add (3) and (4) $\Rightarrow$

(5) $5x^2 - 3x - 3 < 5[x^2] - 3[x] + 2 < 5x^2 - 3x + 5$. Because

$5[x^2] - 3[x] + 2 = 0$, (5) becomes

$$5x^2 - 3x - 3 < 0 < 5x^2 - 3x + 5 \;\Rightarrow\; x \in \left(\frac{3-\sqrt{69}}{10}, \frac{3+\sqrt{69}}{10}\right)$$

It results:
$[x] = -1$ or $[x] = 0$ or $[x] = 1$. For the firsts two values, it is not verify the initial equation.
Hence $[x] = 1 \;\Rightarrow\; x \in [1,2) \;\Rightarrow\; x^2 \in [1,4)$. It results $[x^2] = 1$ or $[x^2] = 2$ or $[x^2] = 3$

The solution is not verify for any of above values.
Correct answer is e.

AL - IX. 009 Find $m \in Z/\{0\}$ for which the equation $\left[\frac{m^2x-1}{2}\right] = \frac{2x+1}{5}$, has solutions and determine the number of solutions.

a) n =2; b) n =3; c) n =4; d) n =5, e) n =1; f) n =0

Solution:

$$\frac{2x+1}{5} = k \text{ and } \frac{2x+1}{5} \le \frac{m^2x-1}{2} < \frac{2x+1}{5} + 1$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x = \frac{5k-1}{2} \quad \text{and} \quad \frac{7}{5m^2-4} \le x < \frac{17}{5m^2-4}$$

$$\Downarrow$$

$$\frac{7}{5m^2-4} \le \frac{5k-1}{2} < \frac{17}{5m^2-4}$$

$$\Downarrow$$

$$(*) \qquad \frac{5m^2+10}{5\left(5m^2-4\right)} \le k < \frac{5m^2+30}{5\left(5m^2-4\right)}$$

Since the left member is positive, it is necessary that

$$\frac{5m^2+30}{5\left(5m^2-4\right)}>1 \Leftrightarrow m^2<\frac{5}{2}=2,5 \text{ to have solutions}$$

The only integer numbers ≠0 which verify the above relation are $m=\pm 1$.
For $m=\pm 1$, $(*)$ becomes:

$$3 \le k < 7 \quad \Leftrightarrow \quad k \in \{3,4,5,6\}$$

We have 4 solutions.
The correct answer is c.

AL - IX. 015 Find the values of real parameter such that

$$\left\{x \in \mathbf{R} : (m-1)x^2-(m+1)x+m+1>0\right\}=\varnothing .$$

a) $m \in (-\infty,-1) \cup \left[\frac{5}{3},+\infty\right)$ b) $m \in [1,+\infty)$ c) $m \in (-\infty,-1]$

d) $m \in \left[\frac{5}{3},+\infty\right)$ e) $m \in \left[-1,\frac{5}{3}\right]$ f) $m \in (-\infty,1]$

Solution:

We impose the conditions: $m-1<0,\ \Delta=(m+1)^2-4\left(m^2-1\right) \Leftrightarrow m<1$

and $m^2+2m+1-4m^2+4 \le 0 \Leftrightarrow m<1$ and $-3m^2+2m+5 \le 0$

$$m_{1,2}=\frac{-1\pm\sqrt{1+15}}{-3}=\frac{-1\pm 4}{-3}$$

Hence $m<1$ and $m \in (-\infty,-1] \cup \left[\frac{5}{3},+\infty\right)$

$\Rightarrow m \in (-\infty,-1]$.

The correct answer is c.

AL - IX. 024 Let us consider the equation $3mx^2+(2m+1)x+m+1=0$, $m\in\mathbf{R}$, with the root x_1 and x_2. Find a relation between the equation's roots, which is independent of m.

a) $x_1+x_2=x_1x_2$ b) $x_1^2+x_2^2=2x_1x_2$ c) $x_1^2-x_2^2=2x_1x_2$

d) $x_1+x_2+x_1x_2=-\dfrac{1}{3}$ e) $x_1^2+x_2^2-3x_1x_2=0$ f) $x_1^2+x_2^2+x_1x_2=0$

Solution:

We will write the Viéte's relations:

$$\begin{cases} x_1+x_2=-\dfrac{2m+1}{3m} \\ x_1x_2=\dfrac{m+1}{3m} \end{cases} \Rightarrow \begin{cases} x_1+x_2=-\dfrac{2}{3}-\dfrac{1}{3m} \\ x_1x_2=\dfrac{1}{3}+\dfrac{1}{3m} \end{cases} + \Rightarrow$$

$$\Rightarrow x_1+x_2+x_1x_2=-\frac{1}{3}$$

The correct answer is d.

AL - IX. 032 Let us consider the polynomial of second degree $f_m(x)=mx^2-(2m-1)x+m-1$, $(m\neq 0)$. Find m such that the peak of the parabola which corresponds to this mapping, it is situated on the first bisectrice.

a) $m=\dfrac{1}{4}$ b) $m=4$ c) $m=\dfrac{1}{2}$ d) $m=2$ e) $m=\dfrac{1}{6}$ f) $m=6$

Solution:

$$f_m(x)=mx^2-(2m-1)x+m-1 \qquad (m\neq 0)$$

$$x_V=-\frac{b}{2a}\Rightarrow x_V=\frac{2m-1}{2m}$$

$$y_V=-\frac{\Delta}{4a}\Rightarrow y_V=-\frac{(2m-1)^2-4m(m-1)}{4m} \Rightarrow \begin{matrix} x_V=\dfrac{2m-1}{2m} \\ y_V=\dfrac{-1}{4m} \end{matrix}$$

$$V\in I\ bis \Rightarrow x_V=y_V \Rightarrow \frac{2m-1}{2m}=-\frac{1}{4m}\Rightarrow \begin{matrix} 8m^2-4m+2m=0 \\ 8m^2-2m=0 \end{matrix}$$

$$\cancel{m=0} \qquad m=\frac{1}{4}$$

The correct answer is a.

AL - IX. 045 Solve the inequation $5x^2 - 20x + 26 \geq \dfrac{4}{x^2 - 4x + 5}$.

a) $[-1,0)$ b) $\left[\dfrac{4}{5},+\infty\right)$ c) $\{0,1\}$ d) **R** e) $\varnothing$ f) $\left(-\sqrt{2},\sqrt{2}\right)$

Solution:

If we designate $x^2 - 4x + 5 = t$ we obtain

$$t \in [-1,0) \cup \left[\frac{4}{5},+\infty\right), \text{ hence}$$

$$-1 \leq x^2 - 4x + 5 < 0 \quad \text{or} \quad x^2 - 4x + 5 \geq \frac{4}{5} \quad \Rightarrow x \in \mathrm{R}$$

The correct answer is d.

AL - IX. 053 Let us consider the equation $mx^2 - x + m - 7 = 0$. Which of the below intervals contains the real parameter *m*, such that the above equation has a unique root in the interval $[2,4]$?

a) $(-\infty,-1]$ b) $(2,+\infty)$ c) $\left(0,\dfrac{1}{2}\right)$ d) $\left[-\dfrac{1}{2},0\right)$ e) $\left[\dfrac{11}{17},\dfrac{9}{5}\right]$ f) $\left(0,\dfrac{9}{5}\right)$

Solution:

We impose the conditions:

$$(1)\quad f(2)\cdot f(4) \leq 0 \qquad \text{şi} \qquad (2)\quad \Delta \geq 0$$

(1) $$(4m - 2 + m - 7)(16m - 4 + m - 7) \leq 0 \Leftrightarrow$$

$$(5m-9)(17m-11) \leq 0 \Leftrightarrow m \in \left[\frac{11}{17},\frac{9}{5}\right] = \mathrm{I}_1$$

(2) $\Delta = 1 - 4m(m-7) \geq 0$ hence: $-4m^2 + 28m + 1 = 0 \quad \Rightarrow m_{1,2} = \dfrac{7 \pm 5\sqrt{2}}{2}$

It results $\Delta \geq 0$ for $m \in \left[\dfrac{7-5\sqrt{2}}{2},\dfrac{7+5\sqrt{2}}{2}\right] = \mathrm{I}_2$

It is necesary that *m* to belong to $I = I_1 \cap I_2$, hence $\Rightarrow m \in \left[\frac{11}{17}, \frac{9}{5}\right]$.

The correct answer is e.

AL - IX. 062 Find $a, b \in \mathbf{Z}^*$ such that the solutions of system

$$\begin{cases} \sqrt[3]{\dfrac{x}{y}} + \sqrt[3]{\dfrac{y}{x}} = a \\ x + y = b \end{cases}$$

be integer numbers.

a) a = 2; b = 1 b) a = 2; b = k c) a = 2; b = 2k
d) a = 2; b = -1 e) a = 4; b = 2k f) a = 2k; b = k

Solution:

If we cube the first equation, it results

$$\frac{x}{y} + \frac{y}{x} + 3\sqrt[3]{\frac{x}{y}}\sqrt[3]{\frac{y}{x}}\left(\sqrt[3]{\frac{x}{y}} + \sqrt[3]{\frac{y}{x}}\right) = a^3$$ we obtain the system

$$\begin{cases} \frac{x}{y} + \frac{y}{x} = a\left(a^2 - 3\right) \\ x + y = b \end{cases} \text{,hence } \begin{cases} x + y = b \\ xy = \dfrac{b^2}{(a-1)^2(a+2)} \end{cases}$$

$$t^2 - bt + \frac{b^2}{(a-1)^2(a+2)} = 0 \quad \Rightarrow \quad t_{1,2} = \frac{b}{2}\left(1 \pm \frac{a+1}{a-1}\sqrt{\frac{a-2}{a+2}}\right)$$

The solutions of system are integer number if $b = 2k, k \in \mathbf{Z}$

and $1 \pm \frac{a+1}{a-1}\sqrt{\frac{a-2}{a+2}} \in \mathbf{Z}$ or $b \in \mathbf{Z}$ and $1 \pm \frac{a+1}{a-1}\sqrt{\frac{a-2}{a+2}} = 2k, \quad k \in \mathbf{Z}$

$$1 \pm \frac{a+1}{a-1}\sqrt{\frac{a-2}{a+2}} = 1 \pm \left(1 + \frac{2}{a-1}\right)\sqrt{\frac{a-2}{a+2}} \in \mathbf{Z} \quad \text{if} \quad a - 1 \in \{-2, -1, 1, 2\} \quad \text{and}$$

$\frac{a-2}{a+2} = n^2, \quad n \in \mathbf{Z} \Rightarrow a = 2, b = 2k, \; k \in \mathbf{Z} \qquad \left(1 \pm \frac{a+1}{a-1}\sqrt{\frac{a-2}{a+2}}\right) = 1$

for a=2 , so it remains only the first case.

The correct answer is c.

AL - IX. 083 Let us consider the inequation $\sqrt{4-x^2} > 1-x$. Which of the below intervals is the set of inequation's solution?

a) $(-\infty,-3)$ b) $\left(\frac{17}{2},20\right)$ c) $(-2,2]$ d) $(22,+\infty)$ e) $[4,5)$ f) $\left(\frac{1-\sqrt{7}}{2},2\right]$

Solution:

We impose condition $4-x^2 \geq 0 \Rightarrow x \in [-2,2]$

Case I $\quad 1-x \leq 0 \Rightarrow [1,\infty)$

Solution (1) $\quad [-2,2] \cap [1,\infty) = [1,2]$

Case II $\quad 1-x > 0 \quad x \in (-\infty,1)$

In this situation we square the inquality

$$4-x^2 > 1-2x+x^2 \quad \Rightarrow x \in \left(\frac{1-\sqrt{7}}{2}, \frac{1+\sqrt{7}}{2}\right)$$

Solution 2 $\quad [-2,2] \cap (-\infty,1) \cap \left(\frac{1-\sqrt{7}}{2}, \frac{1+\sqrt{7}}{2}\right) = \left(\frac{1-\sqrt{7}}{2}, 1\right)$

The final solution = Sol (1) $\cup$ Sol (2) = $[1,2] \cup \left(\frac{1-\sqrt{7}}{2}, 1\right) = \left(\frac{1-\sqrt{7}}{2}, 2\right]$

The correct answer is f.

AL - IX. 085 Solve the equation $x^2+\left(\frac{x}{x-1}\right)^2=1$.

a) $x=1\pm\sqrt{2}$ b) $x=\sqrt{2}\pm1$ c) $x=1-\sqrt{2}\pm\frac{1}{2}\sqrt{2\sqrt{2}-1}$

d) $x=\frac{1-\sqrt{2}}{2}\pm\sqrt{2\sqrt{2}-1}$ e) $x=\pm\frac{1}{2}\sqrt{2\sqrt{2}-1}$ f) $x=\frac{1}{2}\left(1-\sqrt{2}\pm\sqrt{2\sqrt{2}-1}\right)$

Solution:

We add in both members $2x\left(\frac{x}{x-1}\right)$

$$\left(x^2+\left(\frac{x}{x-1}\right)^2+2x\left(\frac{x}{x-1}\right)=1+2x\left(\frac{x}{x-1}\right)\right)\Leftrightarrow$$

$$\Leftrightarrow\left(\left(x+\frac{x}{x-1}\right)^2=1+\frac{2x^2}{x-1}\right)\Leftrightarrow\left(\left(\frac{x^2}{x-1}\right)^2-2\frac{x^2}{x-1}=1\right)$$

We designate $\frac{x^2}{x-1}=y \Leftrightarrow\left(y^2-2y-1=0\right)\Leftrightarrow\left[\begin{array}{l}y=1+\sqrt{2}\\ y=1-\sqrt{2}\end{array}\right.$

$$\left(\frac{x^2}{x-1}=1+\sqrt{2}\right)\Leftrightarrow\left(x^2-x\left(1+\sqrt{2}\right)+1+\sqrt{2}=0\right)\Rightarrow x\in\varnothing$$

$$\left(\frac{x^2}{x-1}=1-\sqrt{2}\right)\Leftrightarrow\left(x^2-x\left(1-\sqrt{2}\right)+1-\sqrt{2}=0\right)\Rightarrow$$

$$\Rightarrow x\in\left\{\frac{1}{2}\left(1-\sqrt{2}\pm\sqrt{2\sqrt{2}-1}\right)\right\}$$

The correct answer is f.

MATEMATICS - X
(symbol AL - X)

AL - X. 009 Find the values of $m \in \mathbf{R}$ such that the equation

$$(3x-1)(x-m-1)^{|x-1|-1} - (2x+m)^{|x-1|} = (x-m-1)^{|x-1|},$$

wish the conditions $x > m+1$ and $x > -\frac{m}{2}$, to have 3 real and distinct roots.

a) $m \in \phi$ b) $m \in \mathbf{R}$ c) $m \in \mathbf{R} \setminus \left\{-\frac{3}{2}, -\frac{1}{2}\right\}$

d) $m \in \left(-\infty, -\frac{2}{3}\right) \setminus \left\{-\frac{3}{2}\right\}$ e) $m \in \left(-\infty, -\frac{1}{2}\right)$ f) $m \in \left(-\frac{1}{2}, \infty\right)$

Solution:

We write $(x-m-1)^{|x-1|=1}(2x+m) - (2x+m)^{|x-1|} = 0$

or $\left(\frac{x-m-1}{2x+m}\right)^{|x-1|-1} = 1$ for $2x+m \neq 0$

and it results $|x-1|-1=0$, hence $x_1 = 0$ $x_2 = 2$

and $\frac{x-m-1}{2x+m} = 1$, hence $x_3 = -2m-1$. The condition $x-m-1>0$ leads to

$-3m-2>0$ hence $m < -\frac{2}{3}$, and $-2m-1 \neq 0$ and $-2m-1 \neq 2$

$\Rightarrow m \in \left(-\infty, -\frac{2}{3}\right) \setminus \left\{-\frac{3}{2}\right\}$, $x_1 = 0 \Rightarrow m > 0$ implies $m \in \varnothing$

The correct answer is a.

AL - X. 031 If we have the expression:

$$E = \sqrt{(\log_x 2 + \log_2 x - 2)\log_2 x} + \sqrt{(\log_x 2 + \log_2 x + 2)\log_2 x},$$

find all the values of $x \in \mathbf{R}$ such that $E = 2$.

a) $[1,+\infty)$ b) $[1,2]\cup\{3\}$ c) $\left[\frac{1}{2},2\right]$

d) $\left[\frac{1}{2},2\right]\setminus\{1\}$ e) $[1,2]\setminus\left\{\frac{3}{2}\right\}$ f) $(1,2)\cup(3,+\infty)$

Solution:

We impose the conditions $x>0,\ x\neq 1 \quad y=\log_2 x \Rightarrow$

$$E=\sqrt{(y-1)^2}+\sqrt{(y+1)^2}=|y-1|+|y+1|,\ (\forall)\, y\in \mathbb{R}\setminus\{0\}$$

$$E=\begin{cases}-2y, & y\in(-\infty,-1)\\ 2, & y\in[-1,1]\\ 2y, & y\in(1,\infty)\end{cases}$$

$$\Rightarrow E=2 \Leftrightarrow \quad y\in[-1,1]\setminus\{0\} \Leftrightarrow x\in\left[\frac{1}{2},2\right]\setminus\{1\}$$

The correct answer is d.

AL - X. 044 Solve in **R** the system: $\begin{cases}x^{\lg y}\cdot y^{\lg z}\cdot z^{\lg x}=10\\ x^{\lg y\lg z}\cdot y^{\lg x\lg z}\cdot z^{\lg x\lg y}=1000\\ xyz=10\end{cases}$.

a) $x=10,\ y=z=1$ b) $x=y=10,\ z=1$ c) $x=y=z=10$

d) $x=y=z=10^{-1}$ e) The system has no solution in **R** f) $x=1,\ y=5,\ z=2$

Solution:

Not. $\lg x=u,\ \lg y=v,\ \lg z=t;\quad x,y,z>0\Rightarrow$

$$\begin{cases}uv+ut+vt=1\\ uvt=1\\ u+v+t=1\end{cases}\Leftrightarrow w^3-s_1w^2+s_2w-s_3=0\Leftrightarrow w^3-w^2+w-1=0$$

$$\Leftrightarrow (w-1)\left(w^2+1\right)=0\Rightarrow \text{ The system has no solution in } \mathbb{R}$$

The correct answer is e.

AL - X. 051 Which is the maximum domain of definition D for the mapping:

$$f : D \to \mathbf{R}\ ,\ f(x) = C_{7x}^{x^2+10} + C_{5x+4}^{x^2+3x-4}\ ?$$

a) $D = \{1,9,11\}$ b) $D = \{2,3,4\}$ c) $D = (-\infty,-1] \cap \mathbf{Z}$
d) $D = [7,+\infty) \cap \mathbf{N}$ e) $D = \{2,3,4,5\}$ f) $D = [1,6] \cap \mathbf{N}$

Solution:

$$C_n^k;\ n,k \in \mathsf{R},\ n \ge k$$

$$x^2+10,\ 7x,\ 5x+4,\ x^2+3x-4 \in \square,\ x \in \mathsf{R}^*$$

$$\begin{cases} 7x \ge x^2+10 \\ 5x+4 \ge x^2+3x-4 \end{cases} \Leftrightarrow \begin{cases} x^2-7x+10 \le 0 \\ x^2-2x-8 \le 0 \end{cases} \Leftrightarrow \begin{cases} x \in [2,5] \\ x \in [-2,4] \end{cases} \Leftrightarrow x \in [2,4] \cap \mathsf{R} = \{2,3,4\}$$

The correct answer is b.

AL - X. 058 Calculate the sum:

$$E = C_n^k + C_{n-1}^k + ... + C_{k+1}^k + C_k^k\ \textbf{,}\ \text{ where } n,k \in \boldsymbol{N}, n \ge k\ .$$

a) $E = C_{n+1}^{k-1}$ b) $E = C_{n+1}^{k+1}$ c) $E = C_{n+1}^{k+2}$ d) $E = C_{n+1}^{k-2}$ e) $E = C_{n+2}^{k+1}$ f) $E = C_{n+2}^{k+2}$

Solution:

For $n \ge k+1$ we have $C_{m+1}^{k+1} = C_m^{k+1} + C_m^k$

If m take the values $n, n-1, ..., k+1$ we will obtain:

$$C_{n+1}^{k+1} = C_n^{k+1} + C_n^k$$
$$C_n^{k+1} = C_{n-1}^{k+1} + C_{n-1}^k$$
$$\dots\dots\dots\dots\dots\dots$$
$$C_{k+1}^{k+1} = C_{k+1}^{k+1} + C_{k+1}^k$$

$$C_{n+1}^{k+1} = C_n^k + C_{n-1}^k + ... + C_{k+1}^k + C_{k+1}^{k+1}$$

But $\quad C_{k+1}^{k+1}=C_k^k$, hence $C_n^k+C_{n-1}^k+\ldots+C_{k+1}^k+C_k^k=C_{n+1}^{k+1}$

The correct answer is b.

AL - X. 068 How many terms which do not contain radicals there exists in the binomial extension $\left(\sqrt[3]{x^2}+\sqrt[4]{x}\right)^{16}$?

a) One term
b) Two terms
c) Three terms
d) None
e) Six terms
f) Four terms

Solution:

The general term is

$$T_{k+1}=C_{16}^k\left(x^{\frac{3}{2}}\right)^{16-k}\left(x^{\frac{1}{4}}\right)^k=C_{16}^k x^{\frac{2(16-k)}{3}+\frac{k}{4}}$$

$$\frac{4(32-2k)+3k}{12}=\frac{128-5k}{12}\in\mathbb{N}\Leftrightarrow k\in[0,16],\ k\in\mathbb{N}\Rightarrow$$

$k=4;\ k=16\Rightarrow$ Two terms do not contain radicals.

The correct answer is b.

AL - X. 079 Calculate

$$E=C_n^0-C_n^2+C_n^4-C_n^6+\ldots+(-1)^k C_n^{2k}+\ldots$$

a) $E=2\cos\frac{n\pi}{4}$
b) $E=\sqrt{2^n}\cos\frac{n\pi}{6}$
c) $E=\sqrt{2^n}\cos\frac{n\pi}{4}$
d) $E=2\sin\frac{n\pi}{4}$
e) $E=\sqrt{2^n}\sin\frac{n\pi}{6}$
f) $E=\sqrt{2^n}\sin\frac{n\pi}{4}$

Solution:

$$(i)^{2k} = \left(i^2\right)^k = (-1)^k \ ;$$

$$(1+i)^n = \left(\sqrt{2}\right)^n \left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = \sqrt{2^n}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

We have:

$$(1+i)^n = C_n^0 + C_n^1 i + C_n^2(-1) + C_n^3(-i) + C_n^4 + C_n^5 i + C_n^6(-1) + ... + C_n^{2k}(i)^{2k} =$$

$$= C_n^0 - C_n^2 + C_n^4 - C_n^6 + ... + (-1)^k C_n^{2k} + i\left(C_n^1 - C_n^3 + C_n^5 + ...\right)$$

$$\Rightarrow E = \sqrt{2^n}\cos\frac{n\pi}{4}$$

The correct answer is c.

AL - X. 087 Let us consider $(a_n)_{n\geq 1}$ a sequence with sum of the first n terms $S_n = n^2 + an + b$, where $a, b \in \mathbf{R}$, for all $n \geq 1$. Find a and b such that the sequence $(a_n)_{n\geq 1}$ to be in arithmetical progression with the first termen equal with 2.

a) $a = 2, b = 3$	b) $a \in \mathbf{R}, b \in (1,2)$	c) $a = 1, b = 0$
d) $a = 2, b = 0$	e) $a = 2, b = 1$	f) $a = 1, b = 2$

Solution:

$$a_1 = 2; \quad S_n = \frac{(2+a_n)n}{2} = n^2 + an + b, (\forall) n \geq 1$$

$$2n + na_n = 2n^2 + 2an + 2b, (\forall)\ n \geq 1$$

$$2n + n\left[a_1 + (n-1)r\right] = 2n^2 + 2an + 2b$$

$$n^2 r + \left(2 + a_1 - r\right)n = 2n^2 + 2an + 2b, (\forall)\ n \geq 1$$

$$\begin{cases} r = 2 \\ a_1 = 2a \\ 2b = 0 \end{cases} \Rightarrow \begin{cases} r = 2 \\ b = 0 \\ a_1 = 2a = 2 \Rightarrow a = 1 \end{cases}$$

The correct answer is c.

AL – X. 095 It is posibile that the numbers 7,8,9 to be elements of a geometrical progression?

a) Yes, in order 7,8,9 with a ratio q<1
b) Yes, in order 9,8,7 with a ratio q<1
c) Yes, in order 7,9,8 with a ratio q<1
d) Yes, in order 8,9,7 with a ratio q<1
e) No.
f) Yes, in order 7,9,8 with a ratio q>1

Solution:

Let $\frac{8}{7} = q^m$ and $\frac{9}{8} = q^n$

It results $\frac{8^n}{7^n} = q^{m+n}$ and $\frac{9^m}{8^m} = q^{m+n}$

We have: $\frac{8^n}{7^n} = \frac{9^m}{8^m} \Rightarrow 7^n \cdot 9^m = 8^{m+n}$

But $8 = 7 + 1$ $\Rightarrow$ 8^{m+n} can't be divisible by 7, so the numbers 7,8,9 can't be the terms of a geometrical progression.
The correct answer is e.

AL – X. 098 Let be $n \in \mathbf{N}$, $n \geq 3$ and $a_1, a_2, \ldots, a_n$ the first n terms of a geometrical progression with $a_k > 0, k = \overline{1,n}$. If $S_1 = \sum_{k=1}^{n} a_k$, $S_2 = \sum_{k=1}^{n} \frac{1}{a_k}$ and $p = a_1 a_2 \ldots a_n$,

Then:

a) $p = \left(\frac{S_1}{S_2}\right)^n$ b) $p = \left(\frac{S_2}{S_1}\right)^n$ c) $p = \sqrt{\left(\frac{S_1}{S_2}\right)^n}$

d) $p=\sqrt[n]{\left(\frac{S_1}{S_2}\right)^2}$ e) $p=S_1^n-S_2^n$ f) $p=\frac{S_1+S_2}{S_1S_2}$

Solution:

$$S_1=a_1\frac{q^n-1}{q-1},\ S_2=\sum_{k=1}^{n}\frac{1}{a_1q^{n-1}}=\frac{1}{a_1}\frac{\left(\frac{1}{q}\right)^n-1}{\frac{1}{q}-1}=\frac{1}{a_1}\frac{q^n-1}{q-1}\cdot\frac{1}{q^{n-1}},$$

$$P=a_1^nq^1q^2\ldots q^{n-1}=a_1^nq^{1+2+\ldots+n-1}=a_1^nq^{\frac{(n-1)n}{2}}$$

$$\Rightarrow\frac{S_1}{S_2}=\frac{a_1\frac{q^n-1}{q-1}}{\frac{1}{a_1}\frac{q^n-1}{q-1}\cdot\frac{1}{q^{n-1}}}=a_1^2q^{n-1}\Rightarrow\left(\frac{S_1}{S_2}\right)^n=a_1^{2n}q^{n(n-1)}$$

$$\Rightarrow P=\sqrt{\left(\frac{S_1}{S_2}\right)^n}$$

The correct answer is c.

AL - X. 106 Determine the degree of the polynomials $f\in\mathbf{Z}[X]$ such that f(7)=5 and f(15)=9.

a) 2 b) such polynomial does not exist c) 3
d) 4 e) 6 f) 8

Solution:

$$f=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$$

$$\underbrace{f(15)-f(7)}_{4}=a_0\left(15^n-7^n\right)+a_1\left(15^{n-1}-7^{n-1}\right)+\ldots+a_{n-1}(15-7)=$$
$$=8k,\quad k\in\mathbb{Z},\ \text{absurd}$$

The correct answer is b.

AL - X. 114 Let $f \in \mathbf{R}[X]$ be an at least 2 degree polynomial. If f gives the remainder 2 by dividing by $X+1$ şi $(X+2)f(X)-Xf(X+3)=1$, determine the remainder of *f* divided by X^2-X-2.

a) $1-X$ *b)* $1+X$ *c)* 1 *d)* 0 *e)* X^2-X-2 *f)* X

Solution:

$$f(-1)=2$$

$f(2)=-1$; resulted from the identity of the division

$f(X)=\left(X^2-X-2\right)Q(X)+mX+n;$ we conclude

$$\begin{matrix} f(-1)=-m+n=2 \\ f(2)=2m+n=-1 \end{matrix} \Rightarrow \begin{cases} m=-1 \\ n=1 \end{cases} \Rightarrow -X+1$$

The correct answer is a.

AL - X. 126 Determine λ and $\mu \in \mathbf{Q}$ such that the bgd of the polynomials $f=2X^3-7X^2+\lambda X+3$ and $g=X^3-3X^2+\mu X+3$ is a second degree polynomial.

a) $\lambda=-1, \mu=2$ b) $\lambda=\mu=0$ c) $\lambda=2, \mu=0$

d) $\lambda=2, \mu=-1$ e) $\lambda=\mu=-1$ f) $\lambda=0, \mu=2$

Solution:

We divide and apply the Euclid's algorithm

$$\begin{array}{r|l} 2X^3-7X^2+\lambda X+3 & X^3-3X^2+\mu X+3 \\ \hline -2X^3+6X^2-2\mu X-6 & 2 \\ \hline / -X^2+(\lambda-2\mu)X-3 & \end{array}$$

$$\begin{array}{l|l} X^3-3X^2+\mu X+3 & X^2-(\lambda-2\mu)X+3 \\ \hline -X^3+(\lambda-2\mu)X^2-3X & X+(\lambda-2\mu-3) \end{array}$$

$$/\ -(\lambda-2\mu-3)X^2+(\mu-3)X-3$$

$$-(\lambda-2\mu-3)X^2+(\lambda-2\mu)(\lambda-2\mu-3)X-(\lambda-2\mu-3)\cdot 3$$

$$/\ \left[(\lambda-2\mu)(\lambda-2\mu-3)+\mu-3\right]X+(12-3\lambda+6\mu)\equiv 0$$

$$\Leftrightarrow \begin{cases}\lambda-2\mu=4\\ (\lambda-2\mu)(\lambda-2\mu-3)+\mu-3=0\end{cases} \Leftrightarrow \begin{cases}\mu=-1\\ \lambda=2\end{cases}$$

The correct answer is d.

AL - X. 130 Determine all the polynomials $P\in\mathbf{R}[X]$, such that $P(x+1)=P(x)+4x^3+6x^2+4x+1$ for any $x\in\mathbf{R}$.

a) $kx^3, k\in\mathbf{R}$ b) x^4+x^3-5 c) $x^4+k, k\in\mathbf{R}$

d) $x^5+k, k\in\mathbf{R}$ e) $k\in\mathbf{R}$ f) $x^4+x+k, k\in\mathbf{R}$

Solution:

$$P(x+1)-P(x)=4x^3+6x^2+4x+1, (\forall)x\in\mathbb{R}\Rightarrow \text{degree } P=4,$$

$$\Rightarrow P(x)=ax^4+bx^3+cx^2+dx+e;$$

$$P(x+1)=P(x)a\left(x^4+4x^3+6x^2+4x+1\right)+b\left(x^3+3x^2+3x+1\right)+$$

$$+c\left(x^2+2x+1\right)+d(x+1)+e-ax^4-bx^3-cx^2-dx-e=$$

$$=4ax^3+(6a+3b)x^2+(4a+3b+2c)x+a+b+c+d\equiv 4x^3+6x^2+4x+1$$

$$\Leftrightarrow \begin{cases}a=1\\ 6a+3b=6\\ 4a+3b+2c=4\\ a+b+c+d=1\end{cases} \Leftrightarrow \begin{cases}a=1\\ b=0\\ c=0\\ d=0\end{cases} \Leftrightarrow P(x)=x^4+k,\quad k\in\mathbb{R}$$

The corect answer c.

AL - X. 131 Let $f \in \mathbf{Z}[X]$ be a polynomial, which four different integer values it is equal to p, p is a prime number. What are the integer values of x such that $f(x)=2p$?

a) does not exist $x \in \mathbf{Z}$ b) for any $x \in \mathbf{N}$ c) for $x = 2k+1,\ k \in \mathbf{Z}$

d) for any $x \in \mathbf{Z}$ e) for $x = 2k,\ k \in \mathbf{Z}$ f) for x to be a prime number

Solution:

$$f(a) = p, f(b) = p, f(c) = p, f(d) = p,$$

$a,b,c,d \in \mathbf{Z}$ different.

$$\Rightarrow f = (X-a)(X-b)(X-c)(X-d)g + p, g \in \mathbf{Z}[X]$$

If $(\exists) X_0 \in \mathbf{Z} : f(X_0) = 2p \Leftrightarrow$

(*) $$(X_0-a)(X_0-b)(X_0-c)(X_0-d)g(X_0) = +p = \text{prime number}$$

Tho equality (*) is impossible because p is a prime number. We conclude that there is no $X_0 \in \mathrm{Z}$ cu $f(x_0) = 2p$

The correct answer is a.

AL - X. 138 Let $P \in \mathbf{R}[X]$, $P = aX^3 + bX^2 + cX + d$, $a,b \neq 0$. Determine the relationship hetween the coefficients a, b, c, d such that the roots of p are in arithmetic progression.

a) $3b^3 + 27ab + 9abc = 0$ b) $2b^3 - 27a^2d + 9abc = 0$ c) $2b^3 + 27a^2d - 9abc = 0$
d) $3a^3 + 27abc - 9bd = 0$ e) $3c^3 + 27abc = 0$ f) $2c^3 + 27a^2d - 9abc = 0$

Solution: We denote x_1, x_2, x_3 with: $u-r, u, u+r$;

$$\begin{cases} x_1 + x_2 + x_3 = -\dfrac{b}{a} \\ x_1x_2 + x_1x_3 + x_2x_3 = \dfrac{c}{a} \\ x_1x_2x_3 = -\dfrac{d}{a} \end{cases} \Leftrightarrow$$

$$\begin{cases} u = -\dfrac{b}{3a} \\ 3u^2 - r^2 = \dfrac{c}{a} \\ u\left(u^2 - r^2\right) = -\dfrac{d}{a} \end{cases} \underset{u \text{ și } r}{\overset{\text{We tako out}}{\Rightarrow}} \ 2b^3 + 27a^2 d - 9abc = 0$$

The correct answer is c.

AL - X. 144 Determine $m \in \mathbf{R}$ such that the rocs x_1 , x_2 , x_3 of the ecuation $x^3 + 2x^2 - mx + 1 = 0$ satisly the relation $x_1^4 + x_2^4 + x_3^4 = 24$.

a) $m = 0, m = -1$ b) $m = 1, m = -1$ c) $m = 0, m = 1$

d) $m = 0, m = -8$ e) $m = -1, m = 3$ f) $m = 4, m = 0$

Solution:

$$\left(x_1^3 + x_2^3 + x_3^3\right) = \left(x_1 + x_2 + x_3\right)^3 - 3\left(x_1 + x_2 + x_3\right)\left(x_1x_2 + x_1x_3 + x_2x_3\right) +$$
$$+3x_1x_2x_3 = \left(-2\right)^3 - 3\left(-2\right)\left(-m\right) - 3 = -6m - 11$$

$$\left.\begin{aligned} x_1^4 &= -2x_1^3 + mx_1^2 - x_1 \\ x_2^4 &= -2x_2^3 + mx_2 - x_2 \\ x_3^4 &= -2x_3^3 + mx_3 - x_3 \end{aligned}\right\} \Rightarrow x_1^4 + x_2^4 + x_3^4 = -2\left(-6m - 11\right) + m\left(4 + 2m\right) + 2 = 2m^2 + 16m + 24$$

$$\Rightarrow 2m^2 + 16m + 24 = 24 \quad \Leftrightarrow \quad m = 0, m = -8$$

The correct answer is d.

AL – X. 180 Let $z_1, z_2 \in \mathbf{C}$ and $x + iy = \dfrac{z_1 + z_2}{z_1 - z_2}$. We have:

a) $x = \dfrac{|z_1|^2 + |z_2|^2}{|z_1 - z_2|^2}$, $y = \dfrac{|z_1 z_2|^2}{|z_1 - z_2|^2}$ b) $x = \dfrac{z_1^2 + z_2^2}{z_1^2 - z_2^2}$, $y = i\dfrac{2z_1\overline{z_2}}{z_1^2 - z_2^2}$

c) $x=\frac{|z_1|^2+|z_2|^2}{|z_1+z_2|^2}$, $y=i\frac{z_1\overline{z_2}+\overline{z_1}z_2}{|z_1+z_2|^2}$ d) $x=\frac{|z_1|^2-|z_2|^2}{|z_1-z_2|^2}$, $y=i\frac{z_1\overline{z_2}-\overline{z_1}z_2}{|z_1-z_2|^2}$

e) $x=\frac{|z_1|^2-|z_2|^2}{|z_1-z_2|^2}$, $y=\frac{\overline{z_1}z_2-z_1\overline{z_2}}{|z_1-z_2|^2}$ f) $x=\frac{|z_1|^2-|z_2|^2}{|z_1-z_2|^2}$, $y=\frac{|z_1z_2|^2}{|z_1-z_2|^2}$

Solution:

$$\frac{z_1+z_2}{z_1+z_2}=\frac{(z_1+z_2)(\overline{z_1}-\overline{z_2})}{|z_1-z_2|^2}=\frac{z_1\overline{z_1}-z_2\overline{z_2}+\overline{z_1}z_2-z_1\overline{z_2}}{|z_1-z_2|^2}=$$

$$=\frac{|z_1|^2-|z_2|^2}{|z_1-z_2|^2}+\frac{\overline{z_1}z_2-z_1\overline{z_2}}{|z_1-z_2|^2}$$

$$\underset{X+Y_i}{\overset{\parallel}{Z}}=\overline{z_1}z_2-z_1\overline{z_2}\Rightarrow\overline{Z}=z_1\overline{z_2}-\overline{z_1}z_2=-Z\Rightarrow X-Y_i=-X-Y_i\Rightarrow X=0,Y\in\mathbb{R}$$

$$\Rightarrow Z=Yi\Rightarrow -iZ=Y$$

The correct answer is d.

AL - X. 189 Let z be a complex number such that $|z-a|=\sqrt{a^2-b^2}$, where, $a>b>0$. Compute $\left|\frac{b-z}{b+z}\right|$.

a) a b) $\sqrt{1-\frac{b}{a}}$ c) $\sqrt{\frac{a-b}{a+b}}$ d) $\sqrt{\frac{a^2-b^2}{a^2+b^2}}$ e) $\sqrt{1+\frac{b}{a}}$ f) $\frac{a-\sqrt{b}}{a+\sqrt{b}}$

Solution:

$$|z-a|^2=(z-a)\left(\overline{z-a}\right)=(z-a)\left(\overline{z}-a\right)=z\cdot\overline{z}-a\left(z+\overline{z}\right)+a^2=$$

$$|z|^2-2a.\operatorname{Re}z+a^2=a^2-b^2\Rightarrow\quad |z|^2=2a.\operatorname{Re}z-b^2$$

$$\left|\frac{b-z}{b+z}\right| = \left|\frac{(b-z)(b+\bar{z})}{(b+z)(b+\bar{z})}\right| = \left|\frac{b^2 - z\bar{z} - b(z-\bar{z})}{b^2 + b(z-\bar{z}) + z\bar{z}}\right| =$$

$$= \frac{\left|b^2 - |z|^2 - 2ib\operatorname{Im} z\right|}{\left|2(a+b)\operatorname{Re} z\right|} = \frac{\left|b^2 - a\operatorname{Re} z - ib\operatorname{Im} z\right|}{|\operatorname{Re} z|\cdot|a+b|} =$$

$$\frac{\sqrt{\left(b^2 - a\operatorname{Re} z\right)^2 + b^2(\operatorname{Im} z)^2}}{|\operatorname{Re} z|\cdot(a+b)} = \frac{\sqrt{(\operatorname{Re} z)^2\left(a^2-b^2\right)}}{|\operatorname{Re} z|(a+b)} = \sqrt{\frac{a-b}{a+b}}$$

The correct answer is c.

AL – X. 207 Write the complex number in a trigonometrical form :
$z = 1 + \cos\alpha - i\sin\alpha$, where $\alpha \in (0,\pi)$.

a) $z = 2\cos\frac{\alpha}{2}\left[\cos\left(-\frac{\alpha}{2}\right) + i\sin\left(-\frac{\alpha}{2}\right)\right]$

b) $z = \cos\alpha\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$

c) $z = 4\cos\frac{\alpha}{2}(\cos\alpha + i\sin\alpha)$

d) $z = \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}$

e) $z = \cos\alpha(\cos\alpha + i\sin\alpha)$

f) $z = 2\cos\alpha\left(\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\right)$

Solution:

We use the formulae $1 + \cos\alpha = 2\cos^2\frac{\alpha}{2}$ and $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

We have:

$$Z = 2\cos^2\frac{\alpha}{2} - i2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\right] =$$

$$= 2\cos\frac{\alpha}{2}\left[\cos\left(-\frac{\alpha}{2}\right) + i\sin\left(-\frac{\alpha}{2}\right)\right]$$

The correct answer is a.

AL – X. 216 Let ω be a complex root of the ecuation: $z^n = 1$, $n \in N^*$, $n \geq 2$. What is the value ot the expression: $S = 1 + 2\omega + 3\omega^2 + ... + n\omega^{n-1}$.

a) $S = \frac{1}{\omega - 1}$ b) $S = \frac{1}{1 - \omega}$ c) $S = \frac{n}{\omega - 1}$

d) $S = \frac{n}{1 - \omega}$ e) $S = n \cdot \omega$ f) $S = \frac{n\omega}{\omega - 1}$

Solution:

We have $\omega^n = 1$ şi $1 + \omega + \omega^2 + ... + \omega^{n-1} = 0$
multiplying the given relation by $1 - \omega$. We have

$$(1-\omega)S = 1 + 2\omega + 3\omega^2 + ... + (n-1)\omega^{n-2} + n\omega^{n-1}$$
$$-\omega - 2\omega^2 - ... - (n-1)\omega^{n-1} - n\omega^n$$

We have $$(1-\omega)S = 1 + \omega + ... + \omega^{n-1} - n\omega^n = -n$$

$$(1-\omega)S = -n$$

$$S = \frac{n}{\omega - 1}$$

The correct answer is c.

MATEMATICS - XI
HINGHER ALGEBRA
(symbol AL - XI)

AL - XI. 011 What is the value of the parameter $a \in \mathbf{R}$ for which there exists $x,y,z,t \in \mathbf{R}$, not all of them equal to zero, surch that

$$x\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + y\begin{pmatrix} -2 & -1 \\ 1 & a-1 \end{pmatrix} + z\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} + t\begin{pmatrix} -1 & -3 \\ 1 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ ?$$

a) $a = 1$ b) $a = 0$ c) $a = -1$ d) $a = 2$ e) $a = -2$ f) $a = 4$

Solution:

By identifying the matrices

$$\begin{cases} x - 2y + z - t = 0 \\ 2x - y + 3z - 3t = 0 \\ x + y + z + t = 0 \\ 2x + (a-1)y + 2z + at = 0 \end{cases} \Rightarrow \begin{vmatrix} 1 & -2 & 1 & -1 \\ 2 & -1 & 3 & -3 \\ 1 & 1 & 1 & 1 \\ 2 & a-1 & 2 & a \end{vmatrix} = 0 \Rightarrow a = 0$$

The correct answer is b.

AL - XI. 035 Given the matrix $A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$. Prove that A^n, $n \geq 1$ has the form

$$\begin{pmatrix} 1 & a_n & b_n \\ 0 & 1 & a_n \\ 0 & 0 & 1 \end{pmatrix}$$ and determine a_n and b_n.

a) $a_n = \frac{n}{2}$, $b_n = \frac{n(n+1)}{6}$

b) $a_n = \frac{n}{2}$, $b_n = \frac{n(2n+5)}{12}$

c) $a_n = \frac{n+1}{2}$, $b_n = \frac{n(2n+1)}{6}$

d) $a_n = \frac{n}{2}$, $b_n = \frac{n(3n+5)}{24}$

$a_n = 2n+3$, $b_n = 3n+7$ f) $a_n = \frac{2n+1}{4}$, $b_n = \frac{n(5n+4)}{4}$

Solution:

$$A^{n+1} = A^n \cdot A \Leftrightarrow a_{n+1} = a_n + \frac{1}{2};\ b_{n+1} = b_n + \frac{1}{2} + \frac{1}{3}$$

$$a_1 = \frac{1}{2}, b_1 = \frac{1}{3} \Rightarrow a_n = \frac{n}{2};\quad b_n = \frac{1}{4}\left[1+2+\ldots+(n-1)\right] + \frac{n}{3} = \frac{n(n-1)}{8} + \frac{n}{3}$$

$b_n = \frac{n(3n+5)}{24}$. Indeed,

$$a_1 = \frac{1}{2} \qquad \text{şi}$$

$$a_2 = a_1 + \frac{1}{2} \qquad b_2 = b_1 + \frac{1}{4} + \frac{1}{3}$$

$$a_3 = a_2 + \frac{1}{2} \qquad b_3 = b_2 + \frac{2}{4} + \frac{1}{3}$$

$$\ldots\ldots \qquad \ldots\ldots$$

$$a_n = a_{n-1} + \frac{1}{2} \qquad b_n = b_{n-1} + \frac{n-1}{4} + \frac{1}{3}$$

$$a_n = \frac{n}{2} \qquad b_n = \frac{1}{4}\left(1+2+\ldots+(n-1) + \frac{n}{3}\right)$$

The correct answer is d

AL - XI. 042 Determine the values for the real parameters α şi β such that the matrix:

$$A = \begin{pmatrix} \beta & 1 & 2 & 4 \\ 1 & \alpha & 2 & 3 \\ 1 & 2\alpha & 2 & 4 \end{pmatrix} \text{ has rank 2.}$$

a) $\alpha = 1, \beta = 1$ b) $\alpha = \frac{1}{2}, \beta = 1$ c) $\alpha = 1, \beta = \frac{1}{2}$

d) $\alpha = -\frac{1}{2}, \beta = 1$ e) $\alpha = -1, \beta = \frac{1}{2}$ f) $\alpha = -\frac{1}{2}, \beta = -\frac{1}{2}$

Solution:

A second order determinant formed from the matrix A has to be different then zero and all the third order determinants of A, have to be equal to zero.

Fie $$\Delta_2 = \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = -2 \neq 0 \Rightarrow \Delta_1 = \begin{vmatrix} \beta & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 2(\beta - 1) = 0$$

$$\Rightarrow \beta = 1; \Delta_2 = \begin{vmatrix} 1 & 2 & 4 \\ \alpha & 2 & 3 \\ 2\alpha & 2 & 4 \end{vmatrix} = -2(2\alpha - 1) = 0 \Rightarrow \alpha = \frac{1}{2}$$

For those values:

$$\Delta_3 = \begin{vmatrix} \beta & 1 & 4 \\ 1 & \alpha & 3 \\ 1 & 2\alpha & 4 \end{vmatrix} = 0, \quad \Delta_4 = \begin{vmatrix} \beta & 1 & 2 \\ 1 & \alpha & 2 \\ 1 & 2\alpha & 2 \end{vmatrix} = 0$$

The correct answer is b.

AL - XI. 048 On which of the variation sets of the real parameters α and β the matrix $\begin{pmatrix} \beta & 1 & 2 & 4 \\ 1 & \alpha & 2 & 3 \\ 1 & 2\alpha & 2 & 4 \end{pmatrix}$ has rank equal to 3?

a) $\alpha \in [-1,1], \beta \in [-1,4]$

b) $\alpha \in \left(-7, \frac{2}{3}\right], \beta \in (0,2)$

c) $\alpha \in \left(0, \frac{3}{4}\right), \beta \in \left(-1, \frac{3}{2}\right)$

d) $\alpha \in \left(-3, \frac{3}{5}\right), \beta \in (0,1)$

e) $\alpha \in \left[-\frac{1}{2}, 1\right), \beta \in \left[\frac{1}{2}, 2\right)$

f) $\alpha \in \left(-\frac{1}{2}, 2\right], \beta \in (0,7]$

Solution:

$$\text{If } \begin{vmatrix} \beta & 1 & 2 \\ 1 & \alpha & 2 \\ 1 & 2\alpha & 2 \end{vmatrix} = 0; \quad \begin{vmatrix} \beta & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & 2 & 4 \\ \alpha & 2 & 3 \\ 2\alpha & 2 & 4 \end{vmatrix} = 0 \iff$$

$$\iff \begin{cases} 2\alpha - 2\alpha\beta = 0 \\ 2(\beta - 1) = 0 \\ 2(1 - 2\alpha) = 0 \end{cases} \Rightarrow \text{ For } \alpha = \frac{1}{2}, \beta = 1, \text{ the matrix has the rank equal to 2.}$$

So the rank is 3 if $\alpha \neq \frac{1}{2}$ and $\beta \neq 1$.

The correct answer is d.

AL - XI. 057 Compute the determinant $\Delta = \begin{vmatrix} x^2 & x & 1 \\ 1 & -y & y^2 \\ y^2 & -xy & x^2 \end{vmatrix}$.

a) $\Delta = (x^2 + y)(1 - xy)(x + y^2)$

b) $\Delta = (x^2 - y)(1 - xy)(x - y^2)$

c) $\Delta = (x^2 - y)(1 - xy)(x + y^2)$

d) $\Delta = (x^2 + y)(1 + xy)(x + y^2)$

e) $\Delta = -(x^2 + y)(1 + xy)(x - y^2)$

f) $\Delta = -(x^2 - y)(1 + xy)(x + y^2)$

Solution:

$$\Delta = \frac{1}{xy} \begin{vmatrix} x^2 y & xy & y \\ x & -xy & xy^2 \\ y^2 & -xy & x^2 \end{vmatrix} = \frac{1}{xy} \begin{vmatrix} x^2 y & xy & y \\ x + x^2 y & 0 & xy^2 + y \\ x^2 y + y^2 & 0 & x^2 + y \end{vmatrix} =$$

$$= -\begin{vmatrix} x(1+xy) & y(1+xy) \\ x^2 y + y^2 & x^2 + y \end{vmatrix} = -(1+xy) \begin{vmatrix} x & y \\ y(x^2 + y) & x^2 + y \end{vmatrix} =$$

$$= -(1+xy)\left(x^2+y\right)\left(x-y^2\right)$$

The correct answer is e.

AL - XI. 061 If a, b, c are the lengths of the sides of a triangle h_a, h_b, h_c are the corresponding heights, what is the value of the determinant: $\Delta = \begin{vmatrix} 1 & a & h_b \cdot h_c \\ 1 & b & h_c \cdot h_a \\ 1 & c & h_b \cdot h_a \end{vmatrix}$?

a) $\Delta = abc$ b) $\Delta = 0$ c) $\Delta = a^2+b^2+c^2$

d) $\Delta = 1$; e) $\Delta = 2abc$ f) $\Delta = \frac{1}{2}(ab+ac+bc)$

Solution:

Because: $S = \frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2}$ We have:

$$\Delta = 4S^2 \begin{vmatrix} 1 & a & \frac{1}{bc} \\ 1 & b & \frac{1}{ac} \\ 1 & c & \frac{1}{ba} \end{vmatrix} = 0$$

The correct answer is b.

AL - XI. 076 Find the solutions of the ecuation $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = 0$.

a) $a, -a, 2a, 3a$ b) $a, -a, 2a, -2a$ c) $a, -a, -a, -3a$

d) $a, a, -a, -3a$ e) $a, a, a, -3a$ f) $a, a, -a, 3a$

Solution:

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix} = (x+3a)\begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}$$

$$(x+3a)(x-a)^3 = 0 \Rightarrow x_1 = -3a,\ x_2 = x_3 = x_4 = a$$

The correct answer is e.

AL - XI. 102 Determine the values of the real parameter *m* such that the following system is compatible

$$\begin{cases} x - my + 1 = 0 \\ 2x + y - m = 0 \\ 3x + (m-1)y + m - 1 = 0 \end{cases}.$$

a) {0,2} b) ∅ c) {1,0} d) {-1,1} e) **R** \{-1,1} f) {3,2}

Solution:

$$A = \begin{pmatrix} 1 & -m \\ 2 & 1 \\ 3 & m-1 \end{pmatrix}; \quad \begin{vmatrix} 1 & -m \\ 2 & 1 \end{vmatrix} = 1 + 2m \neq 0 \quad \text{for} \quad m \neq -\frac{1}{2}$$

$$\Delta_{car} = \begin{vmatrix} 1 & -m & -1 \\ 2 & 1 & m \\ 3 & m-1 & 1-m \end{vmatrix} = \begin{vmatrix} 1 & -m & -1 \\ 0 & 2m+1 & 2+m \\ 0 & 4m-1 & 4-m \end{vmatrix} = 6\left(1-m^2\right) = 0$$

$$\Rightarrow m = 1, \quad m = -1$$

For $m = -\frac{1}{2}$ $\Delta_{princ} = \begin{vmatrix} 2 & 1 \\ 3 & -\frac{3}{2} \end{vmatrix} \neq 0$ Δ_{car} is the same

$\Rightarrow \quad m \in \{-1, 1\}$

The correct answer is d.

AL – XI. 110 Determine the parameters $\alpha, \beta \in \mathbf{R}$

such that the system $\begin{cases} \alpha x + \beta y + z = 1 \\ x + \alpha \beta y + z = \beta \\ x + \beta y + \alpha z = 1 \end{cases}$

has the solutions $x = z = \lambda$, $y = -\frac{1}{2}(1+\lambda)$, $\lambda \in \mathbf{R}$.

a) $\alpha = 2, \beta = 0$ b) $\alpha = -2, \beta = 2$ c) $\alpha = \beta = 1$
d) $\alpha = \beta = -2$ e) $\alpha = -2, \beta \in \mathbf{R}$ f) $\alpha \in \mathbf{R}, \beta = 0$

Solution:
First method. If the system is simple singular (that is rank A=2, where A is the coefficient matrix), then det A = 0 $\Leftrightarrow$

$$\begin{vmatrix} \alpha & \beta & 1 \\ 1 & \alpha\beta & 1 \\ 1 & \beta & \alpha \end{vmatrix} = \beta(\alpha - 1)^2 (\alpha + 2) = 0$$

If $\beta = 0$, then $A = \begin{pmatrix} \alpha & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & \alpha \end{pmatrix}$ and x or z must be a parameter, which is not possible.

If $\alpha = 1$, then $A = \begin{pmatrix} 1 & \beta & 1 \\ 1 & \beta & 1 \\ 1 & \beta & 1 \end{pmatrix}$ and rank A = 1,
which is also not possible.

If $\alpha = -2$, then $A = \begin{pmatrix} -2 & \beta & 1 \\ 1 & -2\beta & 1 \\ 1 & \beta & -2 \end{pmatrix}$, and for $\beta \neq 0$, x or z can be considered as parameter.

If z is considered as a parameter, then:

$$\Delta_x = \begin{vmatrix} -2 & \beta & 1 \\ 1 & -2\beta & \beta \\ 1 & \beta & 1 \end{vmatrix} = 0 \qquad \Leftrightarrow \beta = -2 \neq 0$$

If x is considered as a parameter then

= nec.sec. : $\Delta_z = \begin{vmatrix} \beta & 1 & 1 \\ -2\beta & 1 & \beta \\ \beta & -2 & 1 \end{vmatrix} = 0 \ \Leftrightarrow \beta = -2 \neq 0$

Considering $\alpha = \beta = -2$ it results that $x = z = \lambda$, $y = -\frac{1}{2}(1+\lambda)$ checks the system.

The correct answer is d.

ANALYTICAL GEOMETRY - XI
(symbol GA - XI)

GA - XI. 006 A line passing through the vertex B of the triangle ABC cuts the median[AA']and the edge[AC] in K, respectively I. Determine $m \in \mathbf{R}$ checking $\frac{\text{IC}}{\text{IA}} = m\frac{\text{KA'}}{\text{KA}}$

a) $m = 3$ b) $m = \frac{1}{3}$ c) $m = 1$ d) $m = 2$ e) $m = \frac{3}{2}$ f) $m = \frac{4}{3}$

Solution:

Let's take the line BC, as x-axis, and the orthogonal line on BC passing through A, then A(0,a) B(b,0) C(0,b). The midpoint A' of BC has the coordinates $A'\left(\frac{b+c}{2}, 0\right)$, and an arbitrary line passing through B has the equation (d) $y = \lambda(x-b)$

(AA') $\quad \frac{2x}{b+c} = \frac{y-a}{-a}$ $\qquad (AA') \cap d = \{K\}$ hence $x_k = \frac{(\lambda b + a)(b+c)}{2a + \lambda(b+c)}$

If K divides [AA'] by the ratio s, then

$$x_k = \frac{s\frac{b+c}{2}}{1+s} = \frac{(\lambda b+a)(b+c)}{2a+\lambda(b+c)} \Rightarrow s = \frac{KA}{KA'} = \frac{2(\lambda b+a)}{\lambda(c-b)}$$

$$(AC) \quad \frac{x}{c} = \frac{y-a}{-a} \qquad \text{Denoting } \{J\} = (AC) \cap d \ ,$$

It results that $x_J = \frac{C(\lambda b+a)}{C\lambda + a}$ If

$$\text{s=}\frac{AJ}{JC} \Rightarrow \frac{s'C}{1+s'} = \frac{C(\lambda b+a)}{C\lambda+a} \Rightarrow s' = \frac{\lambda b+a}{\lambda(C-b)} = \frac{AJ}{JC} \Rightarrow \frac{JC}{AJ} = 2\frac{KA'}{KA}$$

The correct answer is d.

GA - XI. 012 Let A(-1,0), B(2,0) be two points in the *xOy* – plane and M(0,α) be a moving point of the plane. One considers the pencil of lines passing through the projections of O on the lines MA and MB. Compute the absolute value of the ratio the lines of the pencil divide the line segment [AB].

a) $\frac{1}{4}$ b) $\frac{1}{2}$ c) $\frac{2}{5}$ d) $\frac{7}{8}$ e) $\frac{1}{3}$ f) $\frac{2}{3}$

Solution: The equation ofMA is

$$(MA) \quad -x + \frac{y}{\alpha} = 1 \Leftrightarrow y = \alpha x + \alpha$$

The equation of MB is:

$$(MB) \quad -\frac{x}{2} + \frac{y}{\alpha} = 1 \Leftrightarrow y = -\frac{\alpha}{2}x + \alpha$$

$$P\begin{cases} y = \alpha x + d \\ y = -\frac{1}{\alpha} \end{cases} \Rightarrow P\left(-\frac{\alpha^2}{\alpha^2+1}, \frac{\alpha}{\alpha^2+1}\right)$$

$$Q\begin{cases} y = -\frac{\alpha}{2}x + \alpha \\ y = \frac{2}{\alpha}x \end{cases} \Rightarrow Q\left(\frac{2\alpha^2}{\alpha^2+4}, \frac{4\alpha}{\alpha^2+4}\right)$$

$$(PQ):\frac{(\alpha^2+1)x+\alpha^2}{\alpha^2+2}=\frac{(\alpha^2+1)y-\alpha}{\alpha}$$

$$\Leftrightarrow \alpha(x+2)-(\alpha^2+2)y=0 \Rightarrow \quad x=-2, \quad y=0$$

The fixed point is N(-2,0). Then $k=\frac{NA}{NB}=\frac{1}{4}$

The correct answer is a.

GA - XI. 031 Write the equation of the straight line passing through the intersection point of the lines:
(d_1) $2x - 3y + 6 = 0$, (d_2) $x + 2y - 4 = 0$ and it is orthogonal on the line passing through P(2,2) and cutting the x-axis at 4 units distance from O.

a) $x + y - 2 = 0$

b) $x - 3y + 4 = 0$

c) $x + y -2 = 0$ şi $x - 3y + 4 = 0$

d) $x - 2y + 4 = 0$ şi $6x + y - 2 = 0$

e) $4x + y - 2 = 0$

f) $x - y + 2 = 0$ şi $3x + y - 2 = 0$

Solution:

The equation of the pencil is $(2+\lambda)x+(2\lambda-3)y+6-4\lambda=0 \quad (1)$

and the equation of a line passing through P is $y-2=m(x-2)$.

Imposing the line to pass through the point of coordinate (4,0) respectively (- 4,0), it results $m=-1$, respectively $m=\frac{1}{3}$. Hence the equations of the two lines are: $(2) \quad x+y-4=0$ and $x-3y+4=0 \quad (3)$.

The orthogonality condition ia:

$$-\frac{2+\lambda}{2\lambda-3}=1 \text{ respectively } -\frac{2+\lambda}{2\lambda-3}=-3 \Rightarrow$$

$\lambda=\frac{1}{3}$ respectively $\lambda=\frac{11}{5}$. Two lines are obtained:

$$(\delta_1) \quad x-y+2=0 \quad (\delta_2) \quad 3x+y-2=0$$

The correct answer is f.

GA - XI. 038 Compute the value of the angle formed by the lines $2x - y - 5 = 0$ and $x - 3y + 4 = 0$, which contains the origin.

a) 30° b) 150° c) 45° d) 135° e) 60° f) 120°

Solution:

The inclinations are: $m_1 = 2, \quad m_2 = \frac{1}{3}$

$$tg\theta = \pm \frac{2 - \frac{1}{3}}{1 + 2\frac{1}{3}} = \pm 1 \quad \Rightarrow \theta = 45^0, \quad \theta = 135^0 \quad \Rightarrow \theta = 45^0$$

The correct answer is c.

GA - XI. 054 A straight line moves by parallelism and intersects the coordinates axes in M and N. Two lines of constant directions go through M and N. What are the locus of the intersection point of the two lines ?

a) The midperpendicular of [MN];
b) a cyrcle;
c) a straight line passing through the origin;
d) a ellipsis;
e) an equilateral hyperbola;
f) a parabola.

Solution:

Let $y = ax + b$ be the equation of the moving line, then $M\left(-\frac{b}{a}, 0\right)$; $N(0, b)$. The equations of the constant direction lines are:

$$\begin{cases} y = m_1\left(x + \frac{b}{a}\right) \\ y - a = m_2 x \end{cases} \quad (1)$$

The equation of the locus is obtained by eliminating the parameter b out of (1)⇒

$$y(a - m_1) + x(m_2 - a)m_1 = 0 \quad (2)$$

The line (2) passes through the origin.
The correct answer is c.

GA - XI. 092 Considering the cyrcle determined by the points A(1,1), B(2,0), C(3,2), compute the length of the tangent segment to the cyrcle passing through the origin.

a) 1 b) 10 c) $\sqrt{\frac{14}{3}}$ d) $\frac{14}{5}$ e) $\frac{13}{4}$ f) $\sqrt{\frac{3}{14}}$

Solution:

The center and the radius of the cyrcle are determined as follows:

$$(x-a)^2+(y-b)^2=r^2$$

$$\begin{cases}(1-a)^2+(1-b)^2=r^2\\(2-a)^2+b^2=r^2\\(3-a)^2+(2-b)^2=r^2\end{cases}\Rightarrow a=\frac{13}{6},\ b=\frac{7}{6},\ r=\frac{\sqrt{50}}{6}$$

therefore $\Omega\left(\frac{13}{6},\frac{7}{6}\right)$, and.

$$O\Omega^2=OT^2+r^2\Rightarrow OT^2=O\Omega^2-r^2$$

$$OT^2=\frac{169}{36}+\frac{49}{36}-\frac{50}{36}=\frac{168}{36}\Rightarrow OT^2=\frac{14}{3}$$

The correct answer is c.

GA - XI. 101 Let C be the cyrcle centred in the origin and having the diameter $\overline{AB}$ laying on the x-axis and of length $2r,\ r\in(0,+\infty)$. Let Γ be another cyrcle tangent to [AB] at N and having a variable center $M\in C$. Find the locus of the intersection point of the line MN and the common chord of C and Γ.

a) $x^2+(y-r)^2=r^2$ b) $(x+r)^2+(y-r)^2=4r^2$ c) $\frac{x^2}{\left(\frac{r}{2}\right)^2}-\frac{y^2}{r^2}-1=0$

d) $\frac{x^2}{r^2}+\frac{y^2}{\left(\frac{r}{2}\right)^2}-1=0$ e) $\frac{x^2}{\left(\frac{r}{2}\right)^2}+\frac{y^2}{r^2}-1=0$ f) $x=\pm r$

Solution:

$$C: \quad x^2 + y^2 = r^2;\ M\left(r\cos t, r\sin t\right),\ t \neq 0;\pi$$

$$\Rightarrow \Gamma : \left(x - r\cos t\right)^2 + \left(y - r\sin t\right)^2 = r^2\sin^2 t$$

$$\begin{cases} \Gamma: \quad x^2 + y^2 - 2rx\cos t - 2ry\sin t = r^2\sin^2 t - r^2 \\ C: \quad -x^2 - y^2 \qquad\qquad\qquad\qquad = -r^2 \end{cases}$$

$$\begin{cases} \omega \text{ common chord}: \quad -2rx\cos t - 2ry\sin t = r\sin^2 t - 2r^2 \\ MN \qquad : x = r\cos t \qquad\qquad = -r^2\cos^2 t - r^2 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} -2x^2 + 2ry\sin t = x^2 + r^2 \Rightarrow \sin t = \dfrac{r^2 - x^2}{2ry} \quad \Big|^2 + \Rightarrow \dfrac{x^2}{r^2} + \dfrac{\left(r^2 - x^2\right)^2}{4r^2y^2} = 1 \Leftrightarrow \\ \cos t = \dfrac{x}{r} \quad \Big|^2 \end{cases}$$

$$\Leftrightarrow \left(x^2 + 4y^2 - r\right)\left(x^2 - r^2\right) = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{x^2}{r^2} + \frac{y^2}{\frac{r^2}{4}} - 1 = 0 \qquad 0 \quad \left(x = \pm r \Leftrightarrow t = 0;\pi\right)$$

$$\frac{x^2}{r^2} + \frac{y^2}{\frac{r^2}{4}} - 1 = 0$$

The correct answer is d.

GA - XI. 102 Let's consider the cyrcle of equation $x^2 + y^2 = r^2$, ans the points P(a,0) and Q(b,0). Let M, N be the ends a variable diameter of the cyrcle. Determine the intersection locus of the lines MP and NQ.

a) $x + by + ra = 0$ b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - r^2 = 0$

c) $x+\frac{2ab}{(a+b)^2}y^2=r^2$

d) $\left(x+\frac{2ab}{a+b}\right)^2+y^2=r^2\frac{(a+b)^2}{(a-b)^2}$

e) $\left(x-\frac{2ab}{a+b}\right)^2+y^2=r^2\frac{(a-b)^2}{(a+b)^2}$

f) $x^2+\left(y-\frac{2ab}{a+b}\right)^2=r^2\frac{(a-b)^2}{(a+b)^2}$

Solution: By noting $M(r\cos\alpha, r\sin\alpha)$, $N(-r\cos\alpha, -r\sin\alpha) \Rightarrow$ the intersection point $R(x,y)$ of MP and NQ is obtained solving the system

$$\begin{array}{l}(MP)\\(NQ)\end{array}\begin{cases}y=\dfrac{r\sin\alpha}{r\cos\alpha-a}(x-a)\\ y=\dfrac{r\sin\alpha}{r\cos\alpha+b}(x-b)\end{cases}\Rightarrow$$

$$r\sin\alpha=\frac{a+b}{a-b}y;\quad r\cos\alpha=\frac{a+b}{a-b}x-\frac{2ab}{a-b}\Rightarrow r^2\sin^2\alpha+r^2\cos^2\alpha=r^2$$

$$\Rightarrow y^2\left(\frac{a+b}{a-b}\right)^2+\left(\frac{a+b}{a-b}\right)^2\left(x-\frac{2ab}{a+b}\right)^2=r^2\quad\Bigg|:\frac{(a+b)^2}{(a-b)^2}\Rightarrow$$

$$\left(x-\frac{2ab}{a+b}\right)^2+y^2=r^2\frac{(a-b)^2}{(a+b)^2}$$

The correct answer is e.

GA – XI. 104 Determine the locus of points from which the tangent to a cyrcle are orthogonal.

a) a hyperbola; b) a parabola; c) an ellipsis

d) a cyrcle; e) a straight line f) an equilateral hiperbola

Solution:

Let $x^2 + y^2 = R^2$, be the equation of the cyrcle, then the orthogonal tangents have the equations:

$$\begin{cases} y = mx + R\sqrt{1+m^2} \\ y = -\frac{1}{m}x + R\sqrt{1+\frac{1}{m^2}} \end{cases} \qquad (1)$$

By equaring (1) : $$\begin{matrix} (y-mx)^2 = R^2(1+m^2) \\ (my-x)^2 = R^2(1+m^2) \end{matrix} \qquad (2)$$

⇒ and eliminating the parameters m, it results the equation of the locus $x^2 + y^2 = 2R^2$ $\quad (3)$

which is a concentric cyrcle of radius $R\sqrt{2}$ (the so called an Mőng's cyrcle).

The correct answer is d.

GA - XI. 108 Let E be the ellipsis of equation $\frac{x^2}{1} + \frac{y^2}{4} - 1 = 0$, let d_n be a line of equation: $y = x + n$, $n \in \mathbf{R}$, and let P , Q be the intersection points of E and d_n . Write the equation of the cyrcle C of diameter [PQ].

a) C: $\left(x - \frac{n}{5}\right)^2 + \left(y + \frac{4n}{5}\right)^2 = \frac{8}{25}(n^2 + 5)$, $n \in \mathbf{R}$

b) C: $\left(x + \frac{n}{5}\right)^2 + \left(y - \frac{4n}{5}\right)^2 = \frac{8}{25}(5 - n^2)$, $n \in (-\sqrt{5}, \sqrt{5})$

c) C: $\left(x - \frac{n}{2}\right)^2 + \left(y - \frac{3n}{5}\right)^2 = \frac{8}{25}(n^2 - 2)$, $n \in \mathbf{R}$

d) C: $(x - n)^2 + (y - 3n)^2 = 25n^2$, $n \in \mathbf{R}$

e) C: $\left(x - \frac{n}{5}\right)^2 + \left(y + \frac{1}{5}\right)^2 = 8n^2$, $n \in [-\sqrt{5}, \sqrt{5}]$

f) C: $x^2 + \left(y - \frac{4n}{5}\right)^2 = \frac{25}{8}n^2$, $n \in \mathbf{R}$

Solution:

$$\begin{cases} 4x^2 + y^2 - 4 = 0 \\ y = x + n \end{cases} \Rightarrow \begin{cases} 5x^2 + 2nx + n^2 - 4 = 0 \\ y = x + n \end{cases}$$

$$x_{1,2} = \frac{-n \pm 2\sqrt{5-n^2}}{5}: \quad y_{1,2} = \frac{4nn \pm 2\sqrt{5-n^2}}{5} \qquad n \in \left(-\sqrt{5}, \sqrt{5}\right)$$

$$\frac{x_1 + x_2}{2} = -\frac{n}{5}: \quad \frac{y_1 + y_2}{2} = \frac{4n}{5} \qquad V = \frac{2}{5}\sqrt{10 - 2n^2}$$

$$\left(x + \frac{n}{5}\right)^2 + \left(y - \frac{4n}{5}\right)^2 = \frac{8}{25}\left(5 - n^2\right), \quad n \in \left(-\sqrt{5}, \sqrt{5}\right)$$

The correct answer is b.

GA – XI. 115 Let's consider the ellipsis of equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, a > b and let F be one of the ellupsis focal point. A certain bisecant passing through F intersects the ellipsis in M and N. Compute the value of $E = \frac{1}{FM} + \frac{1}{FN}$

a) $E = \frac{2a}{b^2}$ b) $E = \frac{a}{b^2}$ c) $E = \frac{a}{2b^2}$

d) $E = \frac{2b}{a^2}$ e) $E = \frac{b}{a^2}$ f) $E = \frac{b}{2a^2}$

Solution: Let $M\left(x_1, y_1\right)$ and $N\left(x_2, y_2\right)$ two points of the ellipsis, then

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad \Rightarrow x^2 - Sx + p = 0 \\ y = m\left(x - c\right) \quad \Rightarrow \begin{cases} y_1 + y_2 = S = m\left(S - 2c\right) \\ y_1 y_2 = P = m^2\left(p - cs + c^2\right) \end{cases} \end{cases}$$

$$E = \frac{1}{FM} + \frac{1}{FN} = \frac{MN}{FM \cdot FN}$$

$$\begin{cases} MN = \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2} = \sqrt{\left(1+m^2\right)\left(S^2 - 4p\right)} = \sqrt{\dfrac{4a^2b^4\left(1+m^2\right)^2}{\left(b^2 + a^2m^2\right)^2}} \\ FM = \sqrt{\left(x_1 - c\right)^2 + y_1^2} = \sqrt{\left(1+m^2\right)\left(x_1 - c\right)^2} \\ FN = \sqrt{\left(x_2 - c\right)^2 + y_2^2} = \sqrt{\left(1+m^2\right)\left(x_2 - c\right)^2} \end{cases} \Rightarrow$$

$$\Rightarrow FM \cdot FN = \left(1+m^2\right)\left|P - CS + c^2\right| = \frac{b^4\left(1+m^2\right)}{b^2 + a^2m^2}$$

$$E = \frac{2a}{b^2}$$

The correct answer is a.

GA - XI. 132 : U is a moving point situated on the tangent in A(a,0) to the elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. The perpendicular in A to (OU) intersects the (Oy) axis in V. Find a, b such that AU·OV=12 and the point P(3,2) is situated on the elipse.

a) $\frac{x^2}{18} + \frac{y^2}{8} - 1 = 0$

b) $\frac{x^2}{36} + \frac{3y^2}{16} - 1 = 0$

c) $\frac{x^2}{16} + \frac{y^2}{12} - 1 = 0$

d) $\frac{x^2}{12} + \frac{y^2}{16} - 1 = 0$

e) $\frac{x^2}{18\left(2-\sqrt{2}\right)} + \frac{y^2}{8\left(2+\sqrt{2}\right)} - 1 = 0$

f) $\frac{x^2}{36\left(2-\sqrt{3}\right)} + \frac{y^2}{16\left(2+\sqrt{3}\right)} - 1 = 0$

Solution:

$$U(a,\lambda)\text{, and } V\begin{cases} y-a=-\dfrac{a}{\lambda}(x-a) \\ x=0 \end{cases}$$

$\Rightarrow V\left(0,\dfrac{a^2}{\lambda}\right)$, $OVAU=a^2=12$ The point B is situated on the elipse thus

$$\frac{9}{a^2}+\frac{4}{a^2}-1=0 \qquad \frac{3}{4}+\frac{4}{b^2}-1=0 \Rightarrow b^2=16$$

$$\Rightarrow \frac{x^2}{12}+\frac{y^2}{16}-1=0$$

The correct answer is d.

GA – XI. 164 On the parabola $y^2=2px$ we consider the point M and its symmetric point M' with respect to the symmetry axis of the parabola.
Find the geometrical place of the intersection between the tangent in M' to the parabola and the parallel through M' to the symmetry axis.

a) $2y^2+3px=0$ b) $y^2+3px=0$ c) $3y^2+2px=0$
d) $3y^2+2px^2=1$ e) $2y^2+3px^2=1$ f) $3y^2-2px^2=1$

Solution:

Let $M(\alpha,\beta)$ be a point situated on the parabola.

$\beta^2=2p\alpha$

The tangent at M has the equation: $\beta y=p(x+\alpha)$

The parallel through M' has the equation

$$\begin{cases} \beta y=p(x+\alpha) \\ y=-\beta \\ \beta^2=2p\alpha \end{cases} \underset{\alpha,\beta}{\overset{elimin}{\Rightarrow}} 3y^2+2px=0 \text{ equation of a parabola}$$

The correct answer is c.

GA - XI. 177 Verify if the straight lines (d_1) and (d_2) defined by:

$$(d_1): x = 3 + t,\ y = -2 + t,\ z = 9 + t, \quad (t \in \mathbf{R})$$

$$(d_2): x = 1 - 2s,\ y = 5 + s,\ z = -2 - 5s, \quad (s \in \mathbf{R})$$

And in the affirmative case write the equations of the perpendicular to the plane defined by (d_1), (d_2) which contains the point $P_0(4,1,6)$

a) They are not coplanar

b) Yes; $\frac{x-4}{-2} = \frac{y-1}{-1} = \frac{z-6}{1}$

c) Yes; $\frac{x-4}{-2} = \frac{y-1}{2} = \frac{z-6}{1}$

d) Yes; $\frac{x-4}{-2} = \frac{y-1}{1} = \frac{z-6}{1}$

e) Yes $\begin{cases} x + 2y - 6 = 0 \\ y - z + 4 = 0 \end{cases}$

f) Yes $\begin{cases} x - 2y - 6 = 0 \\ y + z + 4 = 0 \end{cases}$

Solution:

Both straight lines contain the point $M_0(7, 2, 13)$ so they are coplanar.

The parameters of the perpendicular to the plane defined B_4 d_1, d_2 are

$$\overline{d_1} \times \overline{d_2} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ 2 & 1 & -5 \end{vmatrix} = -6\bar{i} + 3\bar{j} + 3\bar{k}$$

the equations of the perpendicular are:

$$\frac{x-4}{-2} = \frac{y-1}{1} = \frac{z-6}{1}$$

The correct answer is d.

GA - XI. 188 Find the symmetric point of the origin O(0,0,0,) with respect to the plane: $\pi: x + y + z - 1 = 0$

a) $O'\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

b) $O'\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$

c) $O'\left(\frac{2}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

d) $O'\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$

e) $O'\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

f) $O'(3,3,3)$

Solution:

$$\begin{cases}(OP) & x=y=z\\(\pi) & x+y+z=1\end{cases}\Rightarrow P\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

$$O'(x',y',z') \qquad \frac{O+x'}{2}=\frac{1}{3};\quad \frac{O+y'}{2}=\frac{1}{3},\quad \frac{O+z'}{2}=\frac{1}{3}$$

$$O'\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$$

The correct answer is d.

GA - XI. 195 The line (d): $\begin{cases}x-y=0\\ \alpha y-z=0\end{cases}$ is parallel with the plane P if:

a) $P:4x+2y-3z=1;\alpha=2$
b) $P:x+y+z=1;\alpha=0$
c) $P:4x-y+3z=0;\alpha=1$
d) $P:x-y-z=0;\alpha=1$
e) $P:z=0;\alpha=1$
f) $P:x+y=0;\alpha=0$

Solution:

$$d:\begin{cases}x=y\\z=\alpha y\end{cases}\quad;\quad O,A(1,1,\alpha)\in d\Rightarrow \overline{d}=(1,1,\alpha)$$

$$P:4x+2y-3z=1\Rightarrow \overline{N}=(4,2,-3)$$

$$\overline{N}\cdot\overline{d}=4+2-3\alpha=0 \quad \Rightarrow \alpha=2$$

The correct answer is a.

GA - XI. 202 Given the points A(3,-1,0), B(0,-7,3), C(-2,1,-1). Find the value of the parameter α such that the straight line (D) $\begin{cases}2x-3y-3z+9=0\\x+\alpha y+z+1=0\end{cases}$ is parallel to the plane determined by A, B şi C.

a) α=4 b) α=1 c) $\alpha=-\frac{1}{4}$ d) $\alpha=-\frac{1}{3}$ e) α=0 f) α=-3

Solution:

The equation of the plane determined by A, B, C:

$$P_{ABC}:\begin{vmatrix} x & y & z & 1 \\ 3 & -1 & 0 & 1 \\ 0 & -7 & 3 & 1 \\ -2 & 1 & -1 & 1 \end{vmatrix} = 0 \Rightarrow P_{ABC}: y + 2z + 1 = 0$$

$$\Rightarrow A = 0,\ B = 2,\ C = 2$$

The straight line parameters:

$$\begin{pmatrix} 1 & m & n \\ 2 & -3 & -3 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{l} l = -3 + 3\lambda \\ m = -5 \\ n = 2\lambda + 3 \end{array}$$

The parallelism condition $A \cdot l + B \cdot m + C \cdot n = 0 \Rightarrow 4\lambda - 4 = 0 \Rightarrow \lambda = 1$

The correct answer is b.

GA - XI. 204 Given the straight line in the space (D) $\frac{x-1}{1} = \frac{y+3}{8} = \frac{z+2}{8}$ and the point A(0,3,1). Find the distence between A and (D).

a) 2 b) 2,5 c) 2,63 d) 3 e) 5 f) 0

Solution:

From the strainght line equation it follows: $M_0(1,-3,-2) \in D$, $\overline{V_D} = (1,8,8)$

$\Rightarrow \overline{AM_0} = \bar{i} - 6\bar{j} - 3\bar{k}$

The distance between the point A and the straight lime (D) is given by the formula:

$$d = \frac{\left|\overline{AM_0} \times \overline{V_D}\right|}{\left|\overline{V_D}\right|} = \sqrt{\frac{893}{129}} = 2,631$$

because:

$$\overline{AM_0} \times \overline{V_D} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -6 & -3 \\ 1 & 8 & 8 \end{vmatrix} = -24\bar{i} - n\bar{j} + 14\bar{k}$$

$$\Rightarrow \left|\overline{AM_0} \times \overline{V_D}\right| = \sqrt{893}; \quad \left|\overline{V_D}\right| = \sqrt{129}$$

The correct answer is c.

ELEMENTS OF MATEMATICAL ANALYSIS - XI
(symbol AM – XI)

AM - XI. 015 Find the limit of the sequence whose general term is:

$$a_n = \frac{\left[\left(n^2+1\right)\left(n^2-n+1\right)\left(n^2-2n+1\right)\right]^n}{\left(n^2+n\right)^{3n}},\ n \geq 1.$$

a) e^2 b) e^{-6} c) e^{-4} d) e^3 e) e^{-3} f) 1

Solution:

We have

$$a_n = \left[\left(1+\frac{1-n}{n^2+n}\right)^{\frac{n^2+n}{1-n}}\right]^{\frac{1-n}{n^2+n}\cdot n} \cdot \left[\left(1+\frac{1-2n}{n^2+n}\right)^{\frac{n^2+n}{1-2n}}\right]^{\frac{1-2n}{n^2+n}\cdot n} \cdot$$

$$\cdot \left[\left(1+\frac{1-3n}{n^2+n}\right)^{\frac{n^2+n}{1-3n}}\right]^{\frac{1-3n}{n^2+n}\cdot n} \to e^{-1}\cdot e^{-2}\cdot e^{-3} = e^{-6}$$

The correct answer is b.

AM - XI. 020 What is the relation between the parameters a and b such that: $\lim\limits_{n\to\infty}\left(a\sqrt{n+1}+b\sqrt{n+2}+\sqrt{n+3}\right)=0$?

a) $a+b=0$ b) $a+b+1=0$ c) $a+b=1$

d) $a=b=1$ e) $a=1, b=0$ f) $a^2=b^2$

Solution:

The limit is:

$$\lim_{n\to\infty}\left[a\left(\sqrt{n+1}-\sqrt{n+3}\right)+b\left(\sqrt{n+2}-\sqrt{n+3}\right)+\left(a+b+1\right)\sqrt{n+3}\right]=$$

$$=\lim_{n\to\infty}\left(a+b+1\right)\sqrt{n+3}=0 \Leftrightarrow a+b+1=0$$

Correct answer is b.

AM - XI. 029 Evaluate: $\lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{15}+...+\frac{1}{4n^2-1}\right)$.

a) 1 b) 2 c) $\frac{3}{2}$ d) $\frac{1}{2}$ e) $\frac{3}{5}$ f) 3

Solution:

The limit becomes:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{4k^2-1}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)\cdot\frac{1}{2}$$

$$=\lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{2n-1}\right)=\frac{1}{2}$$

The correct answer is d.

AM - XI. 079 Let $f:(0,+\infty)\to\mathbf{R}$, be the function defined as

$f(x)=\left[1+\ln(1+x)+\ln(1+2x)+\ldots+\ln(1+nx)\right]^{1/x}$ for every $x>0$.

Calculate $\lim_{x\to 0} f(x)$.

a) 1 b) 0 c) e^n d) $e^{\frac{n(n+1)}{2}}$ e) $e^{\frac{n(n+1)(2n+1)}{6}}$ f) e^{-n^2}

Solution:

Using the properties of the logarithms we get:

$$f(x)=\left\{\left[1+\prod_{k=1}^{n}\ln n\left(1+kx\right)\right]^{\frac{1}{\prod_{k=1}^{n}\ln(1+kx)}}\right\}^{\frac{1}{x}\prod_{k=1}^{n}\ln(1+kx)}=$$

$$=e^{\prod_{k=1}^{n}\ln(1+kx)^{\frac{1}{x}}}$$

$$\text{Therefore }\lim_{x\to 0} f(x)=e^{\sum_{k=1}^{n}k}=e^{\frac{n(n+1)}{2}}$$

The correct answer is d.

AM - XI. 107 Let there be the function $f:\mathbf{R}\to\mathbf{R}$, given by $f(x)=\begin{cases}2-x, x\in\mathbf{Q}\\ 2x, x\in\mathbf{R}\setminus\mathbf{Q}\end{cases}$.

Determine the set of continuity points of the function f.

a) $\mathbf{R}\setminus\left\{\frac{2}{3}\right\}$ b) $\mathbf{R}$ c) $\mathbf{Q}$ d) $\left\{\frac{2}{3}\right\}$ e) $\varnothing$ f) $\{0\}$

Solution:

We show that the function f has only the point $\frac{2}{3}$ as continuity point.

For this, let $x_0\in \mathrm{R}\setminus\mathrm{Q}$ and $(x_n)_{n\in\mathrm{N}}\subset\mathrm{Q}$ with $x_n \underset{n\to\infty}{\to} x_0$

Then $f(x_n)=2-x_n \underset{n\to\infty}{\to} 2-x_0\neq 2x_0=f(x_0)$, hence f is discontinuous at x_0.

Taking now $x_0\in Q\setminus\left\{\frac{2}{3}\right\}$ and $(x_n)_{n\in\mathrm{N}}\subset\mathrm{R}\setminus\mathrm{Q}$ cu $x_n \underset{n\to\infty}{\to} x_0$, we have

$f(x_n)=2x_n \underset{n\to\infty}{\to} 2x_0\neq 2-x_0=f(x_0)$, hence f is discontinous at x_0

If $x_0=\frac{2}{3}$ then for $(\forall)(x_n)_{n\in\mathrm{N}}$, $x_n \underset{n\to\infty}{\to} x_0$, we have

$f(x_n)\underset{n\to\infty}{\to} f(x_0)=\frac{4}{3}$, whence, by Heine's theorem, f is continous at $x_0=\frac{2}{3}$

The correct answer is d.

AM - XI. 009 Evaluate the limit $L=\lim\limits_{n\to\infty}\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\ldots\cdot\cos\frac{x}{2^n}$,

where $x\in\mathbf{R}\setminus\{0\}$.

a) $L=x\sin x$ b) $L=\frac{\sin x}{x}$ c) $L=\sin x$

d) $L=\frac{\sin x}{2}$ e) $L=2\sin x$ f) $L=\frac{\sin 2x}{2x}$

Solution:

Since $\lim_{n\to\infty} x^n \begin{cases} 0 & x\in(-1,1) \\ 1 & x=1 \\ \infty & x\in(1,\infty) \\ \text{does not exist,} & x\in(-\infty,-1] \end{cases}$

It is obvious that $a_n(x)=\dfrac{1+x^n\left(x^2+4\right)}{x\left(x^n+1\right)}$ is not defined at $x=0$ and $x=-1$

We have $\lim_{n\to\infty} a_n(x)=\lim_{n\to\infty}\dfrac{x^n\left(\dfrac{1}{x^n}+x^2+4\right)}{x^n\left(x+\dfrac{1}{x^n}\right)}=\dfrac{x^2+4}{x}$, for $x\in(-\infty,-1)\cup(1,\infty)$

We also have $\lim_{n\to\infty} a_n(x)=\begin{cases}\dfrac{1}{x}, & x\in(-1,0)\cup(0,1)\\ 3, & x=1\end{cases}$

Thus, we obtain

$$f(x)=,\begin{cases}\dfrac{1}{x}: x\in(-1,0)\cup(0,1)\\ 3, \quad x=1\\ \dfrac{x^2+4}{x}, \quad x\in(-\infty,-1)\cup(1,\infty)\end{cases} \qquad \text{Hence} \quad A=\mathrm{R}\setminus\{0,-1\}$$

It is clear that $f(1-0)=1\neq 5=f(1+0)\neq 3=f(0)$ whence $D=\{1\}$.

The correct answer is b.

AM - XI. 012 Compute

$$L=\lim_{n\to\infty} n^k\left(a^{\frac{1}{n}}-1\right)\left(\sqrt{\frac{n-1}{n}}-\sqrt{\frac{n+1}{n+2}}\right), \text{ for } k\in\mathbf{N},\ a\in\mathbf{R},\ a>0,\ a\neq 1.$$

a) $L=\begin{cases}0, k<3\\ -\ln a, k=3\\ +\infty, k>3\end{cases}$ b) $L=\begin{cases}0, k<3\\ -\infty, k>3\\ -\ln a, k=3\end{cases}$ c) $L=\begin{cases}0, k<3\\ -\ln a, k=3\\ -\infty, k>3 \text{ and } a>1\\ +\infty, k>3 \text{ and } a<1\end{cases}$

d) $L=\begin{cases}+\infty, k\geq 3, a>1\\ 0, k\leq 3\end{cases}$ e) $L=\begin{cases}0, k\leq 3\\ +\infty, k>3\end{cases}$ f) $L=\begin{cases}-\infty, k<3 \text{ and } a<1\\ +\infty, k>3 \text{ and } a>1\\ -\ln a, k=0\end{cases}$

Solution:

By using the inqualities $\frac{2}{x}-1<\left[\frac{2}{x}\right]\leq\frac{2}{x}$, we obtain:

$$\left(\frac{2}{x}-1\right)\frac{x}{3}<\frac{x}{3}\left[\frac{2}{x}\right]\leq\frac{2}{x}\cdot\frac{x}{3}=\frac{2}{3} \quad \text{for x>0, whence} \quad \lim_{\substack{x\to 0\\ x>0}}\frac{x}{3}\left[\frac{2}{x}\right]=\frac{2}{3}$$

$$\frac{2}{x}\cdot\frac{x}{3}\leq\frac{x}{3}\left[\frac{2}{x}\right]<\frac{x}{3}\left(\frac{2}{x}-1\right) \quad \text{for x<0, whence} \quad \lim_{\substack{x\to 0\\ x<0}}\frac{x}{3}\left[\frac{2}{x}\right]=\frac{2}{3}.$$

The correct answer is c.

AM - XI. 039 Calculate: $\lim_{n\to\infty}\frac{1}{3n+1}\sum_{k=1}^{n}\cos\frac{\pi}{2n+k}$.

a) $\frac{1}{2}$ b) 0 c) $\frac{1}{4}$ d) $\frac{1}{3}$ e) 1 f) 2

Solution:

Since: $f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}$ and $f(0)=0$, we have

$$f'(0)=\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\begin{cases}\frac{\frac{1}{2^n}}{\frac{1}{n}}, & \text{for } x=\frac{1}{x}\\ \frac{0}{x}=0 & \text{for } x\neq\frac{1}{n}\end{cases}=0$$

Hence f is differentiable at $x=0$ and $f'(0)=0$

The correct answer is b.

AM - XI. 142 Let $f:\mathbf{R}\to\mathbf{R}$, be the function defined by $f(x)=\min\{x^4,x^5,x^6,x^7\}$. Determine the points at which f is not differentiable.

a) $\{-1,0,1\}$ b) $\{-1,0\}$ c) $\{0,1\}$ d) $\varnothing$ e) $\{-1,1\}$ f) $\{0\}$

Solution:

The function can be writen as

$$f(x)=\begin{cases}x^7, & x\in(-\infty,-1]\cup(0,1]\\ x^5, & x\in(-1,0]\\ x^4, & x\in(1,\infty)\end{cases}, \text{ whence } f'(x)=\begin{cases}7x^6, & x\in(-\infty,-1)\cup(0,1)\\ 5x^4, & x\in(-1,0)\\ 4x^3, & x\in(1,\infty)\end{cases}$$

and $f_s'(-1)=7\neq 5=f_d'(-1)$, $f_s'(0)=f_d'(0)$, $f_s'(1)=7\neq f_d'(1)=4$.

Therefore f is not differentiable at $x=-1$ şi $x=1$

The correct answer is e.

AM – XI. 152 Let there be the function $f : D \subset \mathbf{R} \to \mathbf{R}$, $f(x) = \sqrt{x + 8 - 6\sqrt{x-1}}$. Determine the definition domain D and the set M of the points at which f is not differentiable.

a) $D = [1, \infty)$, $M = \phi$
b) $D = [1,10]$, $M = \{1,10\}$
c) $D = [10, \infty)$, $M = \{10\}$
d) $D = [1, \infty)$, $M = \{1,10\}$
e) $D = [1, \infty) \setminus \{10\}$, $M = \{1\}$
f) $D = [1, \infty)$, $M = \{10\}$

Solution:

Using identity

$$\left(3 - \sqrt{x-1}\right)^2 = 8 + x - 6\sqrt{x-1} \Rightarrow \text{it follows}$$

$$f(x) = \sqrt{\left(3 - \sqrt{x-1}\right)^2} = \left|3 - \sqrt{x-1}\right| \text{ whence } D = [1, \infty)$$

Expliciting the absolute value from the expresion of $f(x)$, we have

$$f(x) = \begin{cases} 3 - \sqrt{x-1}, & x \in [1,10] \\ \sqrt{x-1} - 3, & x \in (10, \infty) \end{cases}, \text{ whence}$$

$$f'(x) = \begin{cases} -\dfrac{1}{2\sqrt{x-1}}, & x \in (1,10) \\ \dfrac{1}{2\sqrt{x-1},} & x \in (10, \infty) \end{cases},$$

$$f_a'(1) = -\infty; \quad f_s'(0) = -\frac{1}{6}; \quad f_a'(10) = \frac{1}{6}. \quad \text{Hence } M = \{1,10\}$$

The correct answer is d.

AM - XI. 159 Find the triplets of real numbers (α, β, χ) for which

The function $f : (0, +\infty) \to \mathbf{R}$, $f(x) = \begin{cases} \ln x, \text{ dacă } x \in (0,1] \\ \alpha x^2 + \beta x + \chi, \text{ dacă } x \in (1, +\infty) \end{cases}$

Is twice differentiable in $(0, +\infty)$?

a) $(1,-1,2)$ b) $\left(-1,2,-\frac{3}{2}\right)$ c) $\left(-1,1,-\frac{3}{2}\right)$

d) $\left(-\frac{1}{2},2,-\frac{3}{2}\right)$ e) $\left(\frac{1}{2},2,-\frac{3}{2}\right)$ f) $\left(\frac{1}{2},-2,\frac{3}{2}\right)$

Solution:

Successively we put the conditions that the function f to be continuous and twice differentiable at 1. to be continous at 1, and twice 1 differentiable at 1.

$$f(1-0)=0,\quad f(1+0)=\alpha+\beta+\gamma \Rightarrow \alpha+\beta+\gamma=0 \qquad (1)$$

$$f'(x)=\begin{cases}\frac{1}{x}, & x\in(0,1) \quad f_s'(1)=1\\ 2\alpha x+\beta, & x\in(1,0) \quad f_d'(2\alpha+\beta)\end{cases} \Rightarrow 2\alpha+\beta=1 \qquad (2)$$

$$f''(x)=\left.\begin{cases}-\frac{1}{x^2}, & x\in(0,1) \quad f_s''(1)=-1\\ 2\alpha & x\in(1,\infty) \quad f_d''2\alpha\end{cases}\right\} \Rightarrow 2\alpha=-1 \qquad (3)$$

$$(1),(2),(3) \Rightarrow \alpha=-\frac{1}{2},\quad \beta=2,\quad \gamma=-\frac{3}{2}$$

The correct answer is d.

AM - XI. 178 Given the function $f:\mathbf{R}\to\mathbf{R}$, $f(x)=x^2e^{-\frac{x}{2}}$, evaluate

$$L=\lim_{n\to\infty} f^{(n)}(0).$$

a) $L=+\infty$ b) $L=4$ c) $L=\frac{1}{2}$ d) $L=0$ e) $L=-\frac{1}{2}$ f) $L=2$

Solution:

Applying the Leibnitz formula, for $f(x)=u(x)\cdot v(x)$, where

$u(x)=e^{-\frac{x}{2}},\quad v(x)=x^2$, we obtain

$$f^{(n)}(x)=u^{(n)}\cdot v+C_n^1\,u^{(n-1)}\cdot v'+C_n^2\,u^{(n-2)}\cdot v''+0$$

$$u^{(n)}=\left(-\frac{1}{2}\right)^n e^{-\frac{x}{2}};\quad v'=2x,\quad v''=2\quad v^{(k)}(x)=0 \text{ for } k>2$$

$$f^{(n)}(x)=\left(-\frac{1}{2}\right)^{n}e^{-\frac{x}{2}}\cdot x^{2}+n\left(-\frac{1}{2}\right)^{n-1}e^{-\frac{x}{2}}\cdot 2x+\frac{n(n-1)}{2}\left(-\frac{1}{2}\right)^{n-2}e^{-\frac{x}{2}}\cdot 2$$

$$f^{(n)}(0)=\frac{(-1)^{n}n(n-1)}{2^{n-2}};\quad L=\lim_{n\to\infty}\frac{(-1)^{n}\left[n(n-1)\right]}{2^{n-2}}=0$$

The correct answer is d.

AM - XI. 190 Given the function $f:\mathbf{R}\setminus\{-3\}\to\mathbf{R}$, $f(x)=\dfrac{2x-1}{x+3}$ and the point $x_0=-3+\dfrac{\sqrt{14}}{2}$. Write the equation of the tangent line to the graph of the function f at the point of abscis x_0.

a) $y=2x+4-2\sqrt{14}$ b) $y=2x+8+2\sqrt{14}$ c) $y=4x+8+2\sqrt{14}$
d) $y=4x+8-2\sqrt{14}$ e) $y=2x+8-2\sqrt{14}$ f) $y=x-4+2\sqrt{14}$

Solution:

The equation of the tangent line to the graph of *f* at the point of abscis x_0 is $y-f(x_0)=f'(x_0)(x-x_0)$. Since $f'(x)=\dfrac{7}{(x+3)^2}$

And $f'(x_0)=2$, then $y-\dfrac{-6+\sqrt{14}-1}{\frac{\sqrt{14}}{2}}=2\left(x+3-\dfrac{\sqrt{14}}{2}\right)$, whence

$$y=2x+8-2\sqrt{14}.$$

The correct answer is e.

AM - XI. 203 Solve the inequation $\ln(x^2+1)>x$.

a) $x\in(0,+\infty)$ b) $x\in(-\infty,1)$ c) $x\in(-\infty,0)$

d) $x\in(1,+\infty)$ e) $x\in(-1,+\infty)$ f) $x\in(-\infty,2)$

Solution:

Taking

$$f(x)=\ln\left(x^2+1\right)-x\text{, we obtain}$$

:

$$f'(x)=\frac{2x}{x^2+1}-1=\frac{-(x-1)^2}{x^2+1}<0,$$

Thus the variation table of *f* is.

x	$-\infty$		0		1		∞
f'		$- - - - - - -$			0	$- - - - -$	
f	∞	□	0	□		□	$-\infty$

Hence $f(x)>0$ for $x\in(-\infty,0)$

The correct answer is c.

AM - XI. 219 Let $f:\mathbf{R}\to\mathbf{R}$, be the function gives by $f(x)=\frac{x^2-ax}{\sqrt{x^2+1}}$ where $a\in\mathbf{R}$.

Determine the value of a for which the function f has an extremum point whose distance to the Oy axis is equal to 2.

a) $a=-11, a=12$ b) $a=-12, a=11$ c) $a=-12, a=12$
d) $a=-4, a=3$ e) $a=1, a=-2$ f) $a=4, a=7$

Solution:

The conditions $f'(-2)=0$ and $f'(2)=0$

Determine the required values of a.

Since $f'(x)=\frac{(2x-a)\left(x^2+1\right)-x^3+ax^2}{\left(x^2+1\right)^{3/2}}=\frac{x^3+2x-a}{\left(x^2+1\right)^{3/2}}$, we obtain

$$-8-4-a=0\Rightarrow a=-12$$

$$8+4-a=0\Rightarrow a=12$$

The correct answer is c.

AM - XI. 226 Let $f : D \subset \mathbf{R} \to \mathbf{R}$, be the function defined by $f(x) = \sqrt{ax^2 + b}$, where D is the existence domain of f and $a, b \in \mathbf{R}$. Find the values of a and b such that D is an interval of length 2 and f has an extremum value equal to 1.

a) $a = 1, b = 1$ b) $a = -4, b = -2$ c) $a = 1, b = -1$

d) $a = 0, b = 2$ e) $a = -1, b = 1$ f) $a = -2, b = 0$

Solution:

We have: $f'(x) = \dfrac{ax}{\sqrt{ax^2 + b}}$; $f'(x) = 0 \Rightarrow x = 0 \Rightarrow f(0) = \sqrt{b} = 1 \Rightarrow b = 1$

For that D to be an interval of minimum length it has to

$$\begin{cases} \Delta > 0 \\ \sqrt{S^2 - 4P} = \left| x_1 - x_2 \right| = 2 \end{cases} \Rightarrow \begin{cases} -4ab > 0 \\ 2\sqrt{\dfrac{-b}{a}} = 2 \end{cases}$$

$$\Rightarrow \sqrt{-\frac{1}{a}} = 1 \Rightarrow a = -1 \text{ and } b = 1$$

The correct answer is e.

AM - XI. 235 Find the set of values of real parameter a such that the function

$f : \mathbf{R} \to \mathbf{R}$, $f(x) = \dfrac{x^2 + ax + 5}{\sqrt{x^2 + 1}}$ to have three dinstinct extremum points.

a) $a \in (-3,3)$ b) $a \in (-2,2)$ c) $a \in \{-2,2\}$

d) $a \in [-2,2]$ e) $a \in (-\infty,2) \cup (2,+\infty)$ f) $a \in \left(-\dfrac{1}{2},7\right)$

Solution:

Equating to zero the derivative: $f'(x) = \dfrac{x^3 - 3x + a}{\left(x^2 + 1\right)\sqrt{x^2 + 1}}$, we have $x^3 - 3x + a = 0$. Separating the real roots, by using Rolle's sequence for the equation: $\varphi(x) = x^3 - 3x + a = 0$ we obtain $\varphi'(x) = 3x^2 - 3 = 0$, whence the following variation table results:

$-\infty$	-1	1	∞
$-$	$a+2$	$a-2$	$+$
	$+$	$-$	

$$\begin{cases} a+2>0 \\ a-2<0 \end{cases} \Rightarrow a \in (-2,2)$$

The correct answer is b.

AM - XI. 237 Determine the values of the real parameter m for which the equation $2\ln x + x^2 - 4x + m^2 - m + 1 = 0$ has a real root greatest than 1.

a) $m \in (10,11)$ b) $m \in (-2,-1]$ c) $m \in (-1,2)$

d) $m \in (2,+\infty)$ e) $m \in (-\infty,-1) \cup (2,+\infty)$ f) $m \in (-\infty,-1)$

Solution: Let

where: $f(x) = 2\ln x + x^2 - 4x + m^2 - m + 1$

$x > 0$

It results: $f'(x) = \frac{2}{x} + 2x - 4 = 0 \Leftrightarrow 2x^2 - 4x + 2 = 0$, whence $f'(x) = 0$ implies $x_{1,2} = 1$.

The Rolle's sequence is

x	0	1	$+\infty$
	$-\infty$	$m^2 - m - 2$	$+\infty$

Putting the condition: $m^2 - m - 2 < 0$ it result $m \in (-1,2)$.

The correct answer is c.

AM – XI. 266 Establish in which of the following intervals is laying the points c, from Lagrange's theorem applied to the function $f:(0,\infty)\to\mathbf{R}$, $f(x)=\ln x$ and to the interval $[1,2]$.

a) $\left(1,\sqrt[3]{2}\right)$ b) $\left(\sqrt[3]{2},\sqrt{2}\right)$ c) $\left(\sqrt{2},\frac{3}{2}\right)$

d) $\left(\frac{3}{2},\frac{7}{4}\right)$ e) $\left(\frac{7}{4},2\right)$ f) $(0,1)$

Solution:

According to Lagrange's theorem it results:

$$\ln 2=\frac{1}{C},\quad C\in(1,2).$$

We have: $e^2<8\Rightarrow 2<3\ln 2\Rightarrow \ln 2>\frac{2}{3}\Rightarrow C<\frac{3}{2}$.

We prove that: $C>\sqrt{2}$ that is $\ln 2<\frac{1}{\sqrt{2}}$. It is sufficient to show that the relation

$$(1)\qquad \ln t<\frac{t-1}{\sqrt{t}},\quad t>1 \text{ holds.}$$

Let $g(t)=\ln t-\frac{t-1}{\sqrt{t}},\quad t>1$. It follows $g'(t)=\frac{1}{t}-\frac{t+1}{2+\sqrt{t}}=$

$=-\frac{\left(\sqrt{t}-1\right)^2}{2+\sqrt{t}}<0,\quad$ for $t>1$, whence g is strictly decreasing function and therefore relation (1) is proved. Substituting in (1) $t=2$, we get what we set out to prove.

The correct answer is c.

AM – XI. 275 Let there be the interval $[x_1,x_2]$ of the length less than or equal to $\frac{\pi}{2}$ with $x_1<-x_2$. Find $c\in(x_1,x_2)$ such that the functions $f(x)=\sin x$ and $g(x)=3\cos x$ satisfy the Cauchy theorem on the specified interval.

a) $\frac{x_1 \pm x_2}{2}$ b) $\frac{x_1 - x_2}{2}$ c) $\frac{x_1 + x_2}{2}$

d) $\frac{x_1 \pm x_2}{3}$ e) $\frac{x_1 - x_2}{3}$ f) $\frac{x_1 + x_2}{3}$

Solution:

According to Cauchy's formula for the functions f and g, we have:

$$\frac{\sin x_2 - \sin x_1}{\cancel{3}\left(\cos x_2 - \cos x_1\right)} = \frac{\cos c}{\cancel{-3}\sin c}$$

$$\frac{2\sin\frac{x_2 - x_1}{2}\cos\frac{x_2 + x_1}{2}}{-2\sin\frac{x_2 - x_1}{2}\sin\frac{x_2 + x_1}{2}} = -\frac{\cos c}{\sin c}$$

The unique solution on $\left(x_1, x_2\right)$ is: $C = \frac{x_1 + x_2}{2}$.

The correct answer is c.

AM - XI. 278 Let $f:[0,1] \to \mathbf{R}$, be the function defined by $f(x) = \frac{1}{x+1}$. Applying Lagrange's theorem to the function for the interval $[0, x]$, the point $c \in (0, x)$, is obtained, where $c = \theta \cdot x$, $0 < \theta < 1$ and $\theta = \theta(x)$. Evaluate: $L = \lim\limits_{\substack{x \to 0 \\ x > 0}} \theta(x)$.

a) $L = 1$ b) $L = 2$ c) $L = \frac{1}{2}$ d) $L = \frac{1}{3}$ e) $L = 0$ f) $L = 3$

Solution:

By Lagrange's theorem: $f(x) - f(0) = xf'(\theta(x))$, with $\theta(x) = \theta \cdot x$ cu $\theta \in (0,1)$

$$\text{Since } f'(x) = -\frac{1}{(x+1)^2} \quad , \quad \forall x \in [0,1]$$

We have: $$\frac{1}{x+1}-1=-1\frac{1}{\left[1+\theta\cdot x\right]^2} \quad ,\forall x\in(0,1)$$

Hence $\theta=\theta(x)=\dfrac{\sqrt{x+1}-1}{x} \quad ,\forall x\in(0,1)$, whence

$$L=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\frac{1}{2}$$

The correct answer is c.

HINGHER ALGEBRA - XII
(symbol AL – XII)

AL - XII. 018 On the set **R** we define the composition low

$$x * y = 2x + y, (\forall)x, y \in \mathbf{R}$$

And we note $x_{n+1} = x_n * x; x_1 = x$, $(\forall)x \in \mathbf{R}$.
Determine the natural number $n \geq 2$ for which $x_{2n} = 8(x_n - x) - x, (\forall)x \in \mathbf{R}$

a) $n \geq 2$ b) $n \in \phi$ c) $n = 6$
d) $n = 4$ e) $n = 2$ f) no answer is correct

Solution:

We have: $x_2 = x * x = 3x = \left(2^2 - 1\right)x$

Let us suppose $x_k = \left(2^k - 1\right)x$ and we prove that:

$$x_{k+1} = \left(2^{k+1} - 1\right)x$$

$$x_k * x = \left[\left(2^k - 1\right)x\right] * x = 2\left(2^k - 1\right)x + x = \left(2^{k+1} - 1\right)x$$

so

$$\left(2^{2n} - 1\right)x = 8\left[\left(2^n - 1\right)x - x\right] - x, \quad \forall x \in \mathbb{R}$$

$$\left(2^{2n} - 1\right)x = \left(8 \cdot 2^n - 8 - 8 - 1\right)x, \quad \forall x \in \mathbb{R}$$

$$2^{2n} - 1 = 8 \cdot 2^n - 17 \quad \Leftrightarrow \quad 2^{2n} - 8 \cdot 2^n + 16 = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(2^n - 4\right)^2 = 0 \quad \Leftrightarrow \quad 2^n = 4 \Leftrightarrow n = 2$$

The correct answer is e.

AL - XII. 025 Let the operation „∗” with real numbers be defined such as:
$a * b = ma + nb + p \quad (\forall)a, b \in \mathbf{R}$. The system of constants m,n,p for which the operation $*$ is associative and non-commutative are:

a) (1,0,0); (0,1,0) b) (1,1,0); (0,1,0) c) (1,1,1); (0,1,0)
d) (1,0,0); (1,1,0) e) (1,0,0); (1,1,1) f) (1,1,1); (1,1,0)

Solution:

$$E_1 = (a*b)*c = (ma+nb+p)*c = m(ma+nb+p)+nc+p$$

$$E_2 = a*(b*c) = a*(mb+nc+p) = ma+n(mb+nc+p)+p$$

from $E_1 \cong E_2 \Rightarrow \begin{cases} m(m-1)=0 & (1) \\ n(1-n=0) & (2) \\ p(m-n)=0 & (3) \end{cases}$

Ec (3) can be satisfied in 2 cases a) m = n but then the operation * is commutative and is not interesting for us, so a; b) p=0 and (1) and (2) had us both to 2 posibilities: m=0 and n=0

m = 1 and n = 1 when * is commutative.

and m=1 and n=0

m=0 and n=1 when * is not commutative in which case it is interesting for us.

So the solution are: (1,0,0) and (0,1,0)

The correct answer is a.

AL - XII. 034 Let then be the set $G = \left\{ X^n \middle| X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, n \in \mathbf{N}^* \right\}$

What is the symmetric of the element X^{1997}, with respect to the operation induced on G by multiplication of the matrices?

a) X b) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

d) I_4 e) $\begin{pmatrix} 1997 & 0 & 0 & 0 \\ 0 & 1997 & 0 & 0 \\ 0 & 0 & 1997 & 0 \\ 0 & 0 & 0 & 1997 \end{pmatrix}$ f) no answer are correct

Solution:

We have:

$$X^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad X^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X^4 = I_4$$

But 1997=4.499+1

$$X^{1997} = \left(X^4\right)^{499} \cdot X = X$$

$$\left(X^{1997}\right)^{-1} = X^3 \left(X^3 \cdot X = XX^3 = I_4\right)$$

The correct answer is c.

AL – XII. 039 For each $n \in \mathbf{N}^*$ we define the function

$$f_n : \mathbf{R} \to \mathbf{R}, \quad f_n(x) = \begin{cases} nx, & x > 0 \\ 0, & x \le 0 \end{cases}.$$

Which is the symmetric of the element f_{2001}, with respect to the compositions of the functions?

a) f_1 b) there is none c) f_{2000} d) f_{2002} e) f_{1000} f) f_{1001}

AL - XII. 039

$$f_n\left(f_m(x)\right) = \begin{cases} nf_m(x), & f_m(x) > 0 \overset{m>0}{\Rightarrow} x > 0 \\ & \Downarrow \\ 0, & f_m(x) \le 0 \quad\quad x \le 0 \end{cases} \quad \square\; f_n \circ f_m = f_{nm}$$

$$f_n \circ f_e \overset{A}{=} f_e \circ f_n = f_n \Leftrightarrow e = 1$$

$$f_{2001} \circ f_n = f_n \circ f_{2001} = f_1 \Leftrightarrow \quad 2001n = 1 \Rightarrow n \notin \square^*$$

The correct answer is b.

AL – XII. 040 We consider the set $M = \left\{a + b\sqrt{2} \,\middle|\, a, b \in \mathbf{Z}\right\}$ having the multiplying operation induced from **R** .

Which is the sufficient condition so that the element $x = a + b\sqrt{2}$ admits an inverse in the set M ?

a) There is no inverse of x in M. b) $a^2 - 2b^2 \ne 0$ c) $a^2 - 2b^2 = \pm 1$

d) $a^2 - 2b^2 = 2$ e) $a^2 - 2b^2 = -2$ f) $a^2 - 2b^2 = 0$

Solution:

The inverse of x in M is the symmetric element of the operation x', that is: $x \cdot x' = 1$ or

$$\left(a+b\sqrt{2}\right)\cdot\left(a'+b'\sqrt{2}\right)=1, \quad aa'+2bb'+\sqrt{2}\left(ab'+ba'\right)=1$$

$$\Rightarrow \begin{cases} aa'+2bb'=1 \\ ba'+ab'=0 \end{cases} \qquad \text{Nec.: } \Delta \neq 0, \quad \begin{vmatrix} a & 2b \\ b & a \end{vmatrix} \neq 0$$

$$\text{or } a^2 - 2b^2 \neq 0 \quad \text{(Condition)}$$

But we still need

and

$$\left.\begin{aligned} a' &= \frac{\begin{vmatrix} 1 & 2b \\ 0 & a \end{vmatrix}}{\Delta} = \frac{a}{\Delta} \in \mathbb{Z} \\ b' &= \frac{1}{\Delta}\begin{vmatrix} a & 1 \\ b & 0 \end{vmatrix} = \frac{-b}{\Delta} \in \mathbb{Z} \end{aligned}\right\} \Rightarrow a^2 - 2b^2 = \pm 1$$

The correct answer is c.

AL - XII. 041 $E = \mathbf{R} \times \mathbf{R}$. For any $t \in \mathbf{R}$, let the function $f_t : E \to E$, be defined by $f_t(x,y) = \left(x + ty + \frac{t^2}{2}, y + t\right), (\forall)(x,y) \in E$ and the set $G = \left\{f_t \mid t \in \mathbf{R}\right\}$ having the operation of composing functions. Which is the symmetric of the element f_{-1} ?

a) $g(x,y) = (x,y)$

b) $g(x,y) = (y,x)$

c) $g(x,y) = (x+y,\ y-1)$

d) $g(x,y) = \left(x - y + \frac{1}{2}, y - \frac{1}{2}\right)$

e) $g(x,y) = \left(x + y + \frac{1}{2}, y + 1\right)$

f) $g(x,y) = \left(x + \frac{y}{2} + \frac{1}{8}, y + \frac{1}{2}\right)$

Solution:

The neutral element is the identical function $1_E = f_0$

$$f_{-1} \circ f_t = f_t \circ f_{-1} = f_0$$

$$\Updownarrow$$

$$f_t\left(x - y + \frac{1}{2}, y - 1\right) = (x, y), \quad \forall (x, y) \in E$$

$$\Updownarrow$$

$$\left(x - y + \frac{1}{2} + t(y-1) + \frac{t^2}{2},\ y - 1 + t\right) = (x, y), \quad \forall x, y \in \mathbb{R}$$

$$\Updownarrow$$

$$\begin{cases} 1 = 1 \\ -1 + t = 0 \\ \frac{1}{2} - t + \frac{t^2}{2} = 0 \\ -1 + t = 0 \end{cases} \quad \Rightarrow \quad t = 1 \Rightarrow f_1, \text{ so}$$

$$g(x, y) = \left(x + y + \frac{1}{2}, y + 1\right);$$

The correct answer is e.

AL - XII. 051 We define on **C** the low ’’$*$’’ : $z_1 * z_2 = z_1 \cdot z_2 + i(z_1 + z_2) - 1 - i$.
Determine the neutral element, the symetrizable elements and determine $\alpha \in \mathbf{C}$, so that $(\mathbf{C} \setminus \{\alpha\}, *)$ be an abelian group.

a) $e = 1 - i$; $z' = \frac{2 + iz}{z - 1}$; $\alpha = i$

b) $e = 1$; $z' = \frac{1 - z}{z + i}$; $\alpha = -1$

c) $e = 1 + i$; $z' = \frac{1 + z}{2z - i}$; $\alpha = 2$

d) $e = -i$; $z' = \frac{zi + z}{z - 1}$; $\alpha = -2$

e) $e = 2 + i$; $z' = \frac{1}{z}$; $\alpha = 2$

f) $e = 1 - i$; $z' = \frac{2 - iz}{z + i}$; $\alpha = -i$

Solution:

$$z * e = z, \quad \forall z \in \mathbb{C} \qquad * \text{is obvionsly commutative}$$

$$z(e+i)+ie-1-i = z \quad \forall z \in \mathbb{C} \Rightarrow e+i=1$$

$$e = 1-i$$

$$z \cdot z' = 1-i \Rightarrow z' = \frac{2-iz}{z+i}$$

So any $z \in \mathrm{C} \setminus \{-i\}$ is simetrizable so that $(\mathrm{C} \setminus \{-i\}, *)$ is an abelian group $\alpha = -i$

The correct answer is f.

AL – XII. 058 Let be a group and a,b∈ G so that : $a^2 = b^2 = (ab)^2$. Which one of the following equalities is true?

a) $a^4 = b^4 = e$; b) $a^3 = b^3 = e$; c) $a^4 = b^2 = e$;
d) $a^5 = b^5$; e) $a^3 = b^3$; f) $a^6 = b^6 = e$.

Solution:

The equality $a^2 = (ab)^2$ can be also written as $a^2 = abab$ or $a = bab(1)$ The equality $b^2 = (ab)^2$ is written $b^2 = abab \Rightarrow b = aba(2)$. From (1) and (2) we have $ab = (bab)(aba) \Leftrightarrow ab = b(aba)ba \Leftrightarrow ab = b^3 a$

By multiplying on the right a, we get $aba = b^3 a^2$ that is $b = b^3 a^2$, and from this $b^2 a^2 = e$. As $b^2 = a^2$ the last equality can be written $a^4 = e$ or $b^4 = e$

The correct answer is a.

AL – XII. 069 Let $G = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix}; x, y \in \mathbf{Z}_3 \right\}$.

Determine n∈**N** and the relation between $x, y \in \mathbf{Z}_3$ so that $(G_1, \cdot)$, $G_1 \subset G$ make an isomorphic abelian group with the group $(\mathbf{Z}_n, +)$ of the classes of modulo n rests.

a) $x^2 + y^2 = 1; n = 4$; b) $x^2 + y^2 = 1; n = 9$; c) $x^2 + y^2 \neq 0; n = 6$;
d) x,y ∈ $\mathbf{Z}_3$; n = 9; e) x,y∈ $\mathbf{Z}_3$; n = 4; f) $x, y \neq 0; n = 9$.

Solution:

The neutral element is $E=\begin{pmatrix}\hat{1} & 0\\ 0 & \hat{1}\end{pmatrix}$. The $X=\begin{pmatrix}\hat{x} & y\\ -y & x\end{pmatrix}$ element has an inverse $X'=\begin{pmatrix}x' & y'\\ -y' & x'\end{pmatrix}$ if and only if $X\cdot X'=X'\cdot X=E$ that is $xx'-yy'=1;\quad xy'+yx'=0$

The second relations is $\frac{x'}{x}=-\frac{y'}{y}=\lambda$. Replacing in the first one $x'=\lambda x;\quad y'=-\lambda y$ we get $\lambda'\left(x^2+y^2=1\right)$ from where a solution is $\lambda'=1$ and $x^2+y^2=1$. The number of elements from G is 9 because $x,y\in\left\{\hat{0},\hat{1},\hat{2}\right\}$. The group $(G,\bullet)$ will contain 4 elements because of the condition $x^2+y^2=1$.

So $(G,\bullet)$ is isomorphic with $(Z_n,+)$, so n=4

The correct answer is a.

AL – XII. 071 Let (G, ·) be the multiplicative group of the matrices that have the form:

$$X=\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix},\ (\text{ a,b,c} \in \mathbf{R}).$$

Determine amony its commutative subgroups the isomorphic subgroup with the additive group of real numbers, (**R**, +) .

a) $\begin{bmatrix}1 & 0 & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}$ b) $\begin{bmatrix}1 & a & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ c) $\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

d) $\begin{bmatrix}1 & a & 0\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}$ e) $\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}$ f) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Solution:

The condition of commutativity $X \cdot X' = X' \cdot X$, when

$$X = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad X' = \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix}, \text{ implies: } ac' = a'c \quad (*)$$

But $(*)$ is not satisfied for any $a, b, c \in \mathbb{R}$ in the cases of subgroups generated by matrices d) and e).

So, the subgroup generated by a), b), c), and f) are commutative.

Now, we define $f : (\mathbb{R}, +) \to (G, \cdot)$ by

$$f(x) = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $$f(x + x') = \begin{bmatrix} 1 & 0 & x+x' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & x' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x) \cdot f(x')$$

And f is a bijection.

The correct answer is c.

AL – XII. 094 Calculate the expression in the field of the classes of modulo 11 rests.

$$E = \left(\frac{\hat{3}}{\hat{4}} + \hat{5} + \frac{\hat{8}}{\hat{3}} \cdot \frac{\hat{7}}{\hat{6}} \right) \cdot \frac{\hat{9}}{\hat{2}}$$

a) $E = \hat{0}$; b) $E = \hat{1}$; c) $E = \hat{2}$; d) $E = \hat{3}$; e) $E = \hat{4}$; f) $E = \hat{5}$.

Solution:

$$\frac{\hat{3}}{\hat{4}} = \hat{3} \cdot \hat{4}^{-1} = \hat{3} \cdot \hat{3} = 9; \quad \frac{\hat{7}}{\hat{6}} = \hat{7} \cdot \hat{2} = \hat{3}; \quad \frac{\hat{9}}{\hat{2}} = \hat{9} \cdot \hat{6}^{1} = 10;$$

$$E = \left(\hat{9} \cdot \hat{5} + \hat{10} \cdot \hat{3} = 9 \right) \cdot \hat{10} = \left(\hat{3} + \hat{8} \right) \hat{10} = \hat{0} \cdot \hat{10} = \hat{0};$$

The correct answer is a.

AL – XII. 097 Let $f_k, k=\overline{1,n}$, be the automorphisms of the field $(\mathbf{C},+,\cdot)$, which have the property that: $f_k(x)=x,\ (\forall)x\in\mathbf{R}$.

Calculate $S(z)=\sum_{k=1}^{n} f_k(z)$.

a) S(z) = 0 b) S(z) = n c) S(z) = Re z
d) S(z) = Im z e) S(z) = 2Re z f) S(z) = 2Im z

Solution:

$$f(z_1+z_2)\overset{1)}{=}f(z_1)+f(z_2) \text{ şi } f(z_1\cdot z_2)\overset{2)}{=}f(z_1)\cdot f(z_2),\quad \forall z_1,z_2\in\mathbb{C};$$

so $f:\mathbb{C}\to\mathbb{C}$, $f(x+yi)\overset{1)}{=}f(x)+f(yi)\overset{2)}{\Rightarrow}f(x)+f(y)f(i)\underset{x\in\mathbb{R}}{\overset{f(x)=x}{=\!=}}x+yf(i)$

$$f(i^2)=f(-1)\overset{f(x)=x}{=\!=}-1;\quad \text{so } f(i)=\pm i\Rightarrow f(x+yi)=x\pm yi$$

$f(i^2)=[f(i)]^2$

$$\Downarrow$$

$$f(z)=z,\quad f(z)=\bar{z}$$

(are morphisms and bijections)

$\Rightarrow S(z)=z+\bar{z}=2\operatorname{Re} z$

The correct answer is e.

AL – XII. 124 In the space $\mathbf{R}^3$ we consider the system of vectors:

$$B=\{\overline{v_1}=(1,0,0),\quad \overline{v_2}=(1,1,0),\quad \overline{v_3}=(1,1,1)\}$$

Verify that sistem of vectors is a base in $\mathbf{R}^3$ and find the coordinates of the vector $\bar{v}=(3,1,-1)$ in this base

a) $\overline{V_B}=(1,0,0)$; b) $\overline{V_B}=(1,2,-1)$ c) $\overline{V_B}=(2,2,-1)$

d) $\overline{V_B}=(2,2,0)$ e) $\overline{V_B}=(0,2,-1)$ f) $\overline{V_B}=(0,0,1)$

Solution:

B is a system of generators, if for $\forall \overline{V}(a,b,c)$ we have $(a=3, b=1, c=-1)$:

$\overline{V}(a,b,c) = \alpha_1\overline{V_1} + \alpha_2\overline{V_2} + \alpha_3\overline{V_3} \Rightarrow (\alpha_1, \alpha_2, \alpha_3)$ the coordinates of the vector $\overline{V}$ in basis B.

$$\Rightarrow \begin{matrix} a = \alpha_1 + \alpha_2 + \alpha 3 \\ b = \quad \alpha_2 + \alpha_3 \\ c = \qquad \alpha_3 \end{matrix} \Rightarrow \begin{cases} \alpha_1 = a - b = 2 \\ \alpha_2 = b - c = 2 \\ \alpha_3 = c = -1 \end{cases}$$

$$\Rightarrow \overline{V_B} = (2, 2, -1)$$

The correct answer is c.

AL – XII. 129 Find $\lambda \in \mathbf{R}$, so that the system of vectors

$$S = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} \lambda & 3 \\ 2-\lambda & \lambda+1 \end{pmatrix} \right\}$$

forms a basis in the real vector space $M_2(\mathbf{R})$ with respect to summation and multiplication with a scalar operations of a matrices.

a) $\lambda = 0$ b) $\lambda \neq -\frac{7}{2}$ c) $\lambda = -\frac{7}{2}$

c) $\lambda \neq 0$ d) $\lambda = 2$ e) $\lambda \neq 2$

AL - XII. 129

The system S is a basis iff it is linearly independent.

So
$$\begin{vmatrix} 1 & 0 & -1 & \lambda \\ 2 & 5 & 0 & 3 \\ 1 & -1 & 3 & 2-\lambda \\ -1 & -1 & -1 & \lambda+1 \end{vmatrix} \neq 0 \Leftrightarrow \begin{vmatrix} 1 & 0 & -1 & 2 \\ -3 & 0 & -5 & 52+8 \\ 2 & 0 & 4 & 1-2\lambda \\ -1 & -1 & -1 & \lambda+1 \end{vmatrix} \neq 0 \Leftrightarrow$$

$$\Rightarrow \begin{vmatrix} 1 & -1 & \lambda \\ -3 & -5 & 5\lambda+8 \\ 2 & 4 & 1-2\lambda \end{vmatrix} \neq 0 \Leftrightarrow \begin{vmatrix} 1 & -1 & 2 \\ 0 & -8 & 82+8 \\ 0 & 6 & 1-4\lambda \end{vmatrix} \neq 0 \Leftrightarrow$$

$$\begin{vmatrix} -8 & 8\lambda+8 \\ 6 & 1-42 \end{vmatrix} \neq 0 \qquad \begin{matrix} -9+32\lambda-48\lambda-48 \neq 0 \Leftrightarrow \\ -16\lambda-56 \neq 0 \Leftrightarrow \lambda \neq -\dfrac{56}{16} = -\dfrac{7}{2} \end{matrix}$$

The correct answer is b.

AL – XII. 136 Determine the matrix associated to the linear mapping $f:\mathbf{R}^2 \to \mathbf{R}^3$, $f(x^1,x^2)=(2x^1,x^1+x^2,-x^2)$ in a pair of bases $B_1=\{(2,1),(1,2)\}\subset \mathbf{R}^2$ and $B_2=\{(2,1,0),(0,1,1),(0,0,2)\}\subset \mathbf{R}^3$.

a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ -1 & -2 \end{bmatrix}$

d) $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 2 \end{bmatrix}$ f) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

Solution: We have:

$$\begin{cases} f(2,1)=a_1^1(2,1,0)+a_1^2(0,1,1)+a_1^3(0,0,2) \\ f(1,2)=a_2^1(2,1,0)+a_2^2(0,1,1)+a_2^3(0,0,2) \end{cases}$$

or

$$\begin{cases} (4,3,-1)=\left(2a_1^1,a_1^1+a_1^2,a_1^2+2a_1^3\right) \\ (2,3,-2)=\left(2a_1^2,a_1^2+a_2^2,a_2^2+2a_2^3\right) \end{cases}$$

$$\Rightarrow \begin{cases} 2a_1^1=4 & \Rightarrow a_1^1=2 \\ a_1^1+a_1^2=3 & \Rightarrow a_1^2=1 \\ a_1^2+2a_1^3=-1 & \Rightarrow a_1^3=-1 \end{cases} \quad \text{şi} \quad \Rightarrow \begin{cases} 2a_2^1=2 & \Rightarrow a_2^1=1 \\ a_2^1+a_2^2=3 & \Rightarrow a_2^2=2 \\ a_2^2+2a_2^3=-2 & \Rightarrow a_2^3=-2 \end{cases}$$

$$A=[f]_{(B_1,B_2)}=\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -2 \end{bmatrix}$$

The correct answer is a.

AL – XII. 140 Let be the linear transformation $f:\mathbf{R}^2 \to \mathbf{R}^2$, $f(x)=(-7x_1+10x_2, -5x_1+8x_2)$, for every $x=(x_1,x_2)\in\mathbf{R}^2$. Find the vector $x\in\mathbf{R}^2$ for which $f(x)=0$ and then search the values of $x\in\mathbf{R}$ for which there exists $x\in\mathbf{R}^2\setminus\{(0,0)\}$ such that $f(x)=\lambda x$.

a) $x=(0,0), \lambda\in\{-2,2\}$
b) $x=(0,0), \lambda\in\{-2,3\}$
c) $x=(0,0), \lambda\in\{2,3\}$
d) $x=(0,0), \lambda\in\{2,-3\}$
e) $x=(0,0), \lambda\in\{-2,-3\}$
f) $x=(0,0), \lambda\in\{-3,2\}$

Solution:

$$f(x)=0 \Leftrightarrow (-7x_1+10x_2, -5x_1+8x_2)=(0,0)\Rightarrow$$

$\begin{cases}-7x_1+10x_2=0\\ -5x_1+8x_2=0\end{cases}$. This homogenous linear system with 2 equation and 2 unknowns admits just the common solution $x_1=0,\quad x_2=0$ because the system determinante is $\Delta=\begin{vmatrix}-7 & 10\\ -5 & 8\end{vmatrix}=-56+50=-6\neq 0$. Therefore $x=(0,0)\in\mathsf{R}^2$.

$$f(x)=\lambda x \Leftrightarrow (-7x_1+10x_2, -5x_1+8x_2)=(\lambda x_1, \lambda x_2)\Leftrightarrow$$

$\begin{cases}(-7-\lambda)x_1+10x_2=0\\ -5x_1+(8-\lambda)x_2=0\end{cases}$ This system admits $\Delta=0$, solution if and only if the system determinante $\Delta=\begin{vmatrix}-7-\lambda & 10\\ -5 & 8-\lambda\end{vmatrix}=0\Leftrightarrow \lambda^2-\lambda-6=0\Rightarrow \quad \lambda_1=-2, \lambda_2=3$

The correct answer is b.

ELEMENTS OF MATHEMATICAL ANALYSIS - XII
(symbol AM - XII)

AM - XII. 001 Let be the function $f:\mathbf{R}\to\mathbf{R}$, $f(x)=\begin{cases}\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}, & x\neq 0\\ k, & x=0\end{cases}$, Find the value of the real parameter k for which *f* admits primitives on **R**.

$k=0$ b) $k=1$ c) $k=0$ on $k=1$ d) $k=2$ e) $k\in\mathbf{R}$ f) does not exist k

Solution:

For $x\neq 0,\ f(x)=\left(x\sin\frac{1}{\alpha}\right)'$. If f has primitives a R, let be $F:\mathrm{R}\to\mathrm{R}$ a primitive of *f*.

Then $F(x)=x\sin\frac{1}{x}+c,\quad \forall x\neq 0, c\in$.

But F is continuous on $\mathrm{R}\Rightarrow F(0)=C$

But F is derivable on $\mathrm{R}\Rightarrow F'(0)=K=\lim\limits_{x-0}\frac{F(x)-F(0)}{x-0}=\lim\limits_{x\to 0}\sin\frac{1}{x}$, an this limit does not exist. Therefore, we obtain a contradiction, so f do not have primitives on R

The correct answer is f.

AM - XII. 004 Let be the function $f:[-1,1]\to\mathbf{R}$, $f(x)=\begin{cases}e^x, & x\in[-1,0)\\ x^2+2, & x\in[0,1]\end{cases}$.

Which from be following assertion are true ?

a) $F(x)=\begin{cases}e^x, x\in[-1,0)\\ 2x, x\in[0,1]\end{cases}$ is a primitive of f

b) $F(x)=\begin{cases}e^{2x}, & x\in[-1,0)\\ 2x+1, x\in[0,1]\end{cases}$ is a primitive of f

c) $F(x)=\begin{cases}e^x+1, x\in[-1,0)\\ \frac{x^3}{3}+2, x\in[0,1]\end{cases}$ is a primitive of f

d) $F(x)=\begin{cases} e^x, & x\in[-1,0) \\ \frac{x^3}{3}+1, & x\in[0,1] \end{cases}$ is a primitive of f

e) f not is a primitive of $[-1,1]$

f) $F(x)=\begin{cases} e^x, & x\in[-1,0) \\ \frac{x^2}{2}+3, & x\in[0,1] \end{cases}$ is a primitive of f

Solution:

f don't have Darboux's property on $[-1,1] \Rightarrow f$ don't have primitive on $[-1,1]$.

$f_{|[-1,0)}$ and $f_{|[0,1]}$ are continous but f is not continous on $[-1,1]$

$$f[-1,1]=\underbrace{\left[\frac{1}{e},1\right)\cup[2,3]}_{\text{Is not interval}}$$

The correct answer is e.

AM – XII. 022 Compute the integral $\int\sqrt{1+tg^2x}dx$, $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

a) $\ln\left(tgx-\frac{1}{\cos x}\right)+C$

b) $\ln\left(tgx+\frac{1}{\cos x}\right)+C$

c) $\ln\left(tgx+\frac{1}{\sin x}\right)+C$

d) $\ln\left(tgx-\frac{1}{\sin x}\right)+C$

e) $\ln(tgx+\cos x)+C$

f) $\ln(tgx-\cos x)+C$

Solution:

With change variable $tgx=t\Rightarrow x=arctgt=\varphi(t)$ we have

$\varphi'(t)=\frac{1}{t^2+1}$ $\qquad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\Rightarrow t\in\mathbb{R}$

$$\int\sqrt{1+tg^2x}dx=\int\sqrt{1+t^2}\cdot\frac{1}{t^2+1}dt=\int\frac{1}{\sqrt{t^2+1}}dt=$$

$$=\ln\left(t+\sqrt{t^2+1}\right)+C=\ln\left(tgx+\frac{1}{\cos x}\right)+C$$

The correct answer is b.

AM - XII. 031 Compute the integral $I=\int \frac{\sin x}{\sin x+\cos x}dx$, where $x\in\left(-\frac{\pi}{4},\frac{3\pi}{4}\right)$.

a) $I=\ln\ \text{tg}\frac{x}{2}+C$

b) $I=\frac{1}{2}\left(x^2-\ln\left|\sin x-\cos x\right|\right)+C$

c) $I=\frac{1}{2}\text{arctg}\, x+C$

d) $I=\frac{1}{2}\left(x-\ln\left|\sin x+\cos x\right|\right)+C$

e) $I=\frac{1}{2}\left(\ln\left|\sin x-\cos x\right|\right)+\text{arctg}\, x+C$

f) $I=\frac{1}{2}\left(x+\ln\left|\sin x+\cos x\right|\right)+C$

Solution:

$$I=\int\frac{\sin x}{\sin x+\cos x}dx;\quad J=\int\frac{\cos x}{\sin x+\cos x}dx$$

$$I+J=\int dx=x+c_1$$

$$J-I=\int\frac{\cos x-\sin x}{\sin x+\cos x}dx=\ln\left|\sin x+\cos x\right|+c_2$$

$$2I=x+c_1-\ln\left|\sin x+\cos x\right|-c_2$$

$$I=\frac{1}{2}\left(x-\ln\left|\sin+\cos x\right|\right)+k$$

The correct answer is d.

AM- XII. 039 Find the primitives of the function

$$f(x)=\sqrt{x+5-4\sqrt{x+1}}+\sqrt{x+10-6\sqrt{x+1}},\quad x\in[3,8].$$

a) $F(x)=\frac{4}{3}\sqrt{(x+1)^3}+C$

b) $F(x)=x+C$

c) $F(x)=\sqrt{x+1}+C$

d) $F(x)=2\sqrt{x+1}+C$

e) $F(x)=\begin{cases}x+C, & x\in[3,5]\\ \frac{4}{3}\sqrt{(x+1)^3}+5-8\sqrt{6}+C, & x\in[5,8]\end{cases}$

f) $F(x)=-5x+C$

Solution:

$$(5+x)^2-16x-16=(x-3)^2,\quad (10+x)^2-36x-36=(x-8)^2$$

$$\Rightarrow f(x)=\sqrt{\frac{x+5+|x-3|}{2}}-\sqrt{\frac{x+5-|x-3|}{2}}+\sqrt{\frac{x+10+|x-8|}{2}}-\sqrt{\frac{x+10-|x-8|}{2}},\quad x\in[3,8]$$

$$f(x)=\sqrt{\frac{2x+2}{2}}-\sqrt{\frac{8}{2}}+\sqrt{\frac{18}{2}}-\sqrt{\frac{2x+2}{2}}=-2+3=1$$

$$\Rightarrow F(x)=x+c$$

The correct answer is b.

AM – XII. 043 Compute the integral:

$$I=\int\frac{\sqrt{1+x^2}}{x}dx,\quad x>0.$$

a) $I=\sqrt{1+X^2}+C$; b) $I=\sqrt{1+x^2}-\ln\frac{\sqrt{1+x^2}+1}{x}+C$;

c) $I=\sqrt{1+x^2}$; d) $I=\sqrt{1+x^2}-\ln\frac{\sqrt{1+x^2}+1}{x}$;

e) $I=x+\sqrt{1+x^2}+C$; f) $I=\sqrt{1+x^2}+\ln\frac{\sqrt{1+x^2}+1}{x}+C$.

Solution:

$$I=\int\frac{\sqrt{1+x^2}}{x}\cdot dx=\int\frac{1+x^2}{x\sqrt{1+x^2}}=\int\frac{dx}{x\sqrt{1+x^2}}+\int\frac{x}{\sqrt{1+x^2}}=$$

$$=J+\sqrt{1+x^2}$$

where

$$J=\int\frac{dx}{x\sqrt{1+x^2}}=\int\frac{\frac{1}{x^2}dx}{\sqrt{\frac{1}{x^2}+1}}=-\int\frac{d\left(\frac{1}{x}\right)}{\sqrt{\frac{1}{x^2}+1}}=-\ln\left(\sqrt{\frac{1}{x^2}+1}+\frac{1}{x}\right)+C$$

$$\Rightarrow I=\sqrt{1+x^2}-\ln\frac{\sqrt{1+x^2}+1}{x}+C$$

The correct answer is b.

AM - XII. 052 Compute the limit of the sequence with general term:

$$a_n = \frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+...+\sqrt{\frac{n}{n+3(n-1)}}\right].$$

a) 0 b) 2 c) 1 d) e e) 3 f) $\frac{1}{2}$

Solution:

$$a_n = \frac{3}{n}\left[1+\frac{1}{\sqrt{1+\frac{3}{n}}}+\frac{1}{\sqrt{1+\frac{6}{n}}}+...+\frac{1}{\sqrt{1+\frac{3(n-1)}{n}}}\right]$$

$$a_n = \frac{3}{n}\sum_{i=0}^{n-1}\frac{1}{\sqrt{1+i\frac{3}{n}}}$$

We choose the function $f:[0,3]\to \mathrm{R}$, $f(x)=\frac{1}{\sqrt{1+x}}$ which is continous, hence it is integrable, the partition $\Delta_{[0,3]}=\left\{0,\frac{3}{n},\frac{6}{n},\frac{9}{n},...,\frac{3(n-1)}{n},3\right\}$, and the points $\varepsilon_i=\left\{0,\frac{3}{n},\frac{6}{n},...,\frac{3(n-1)}{n}\right\}$

$$\lim a_n = \int_0^3 \frac{dx}{\sqrt{1+x}} = 2\sqrt{1+x}\Big|_0^3 = 2\left(\sqrt{1+3}-\sqrt{1+0}\right)=2$$

The correct answer is b.

AM - XII. 066 Compute the integral $F(a)=\int_0^1 \left|x^2+a\right|dx,\ a\in \mathbf{R}$.

a) $F(a)=\begin{cases} a+\frac{1}{3}, & a\le 0\\ a-\frac{1}{3}, & a>0\end{cases}$

b) $F(a)=\begin{cases} -a-\frac{1}{3}, & a\le -1\\ -\frac{4}{3}a\sqrt{-a}+a+\frac{1}{3}, & -1<a\le 1\\ a-\frac{1}{3}, & 1<a\end{cases}$

c) $F(a)=\begin{cases}-a-\frac{1}{3}, & a\le -1\\ -\frac{4}{3}a\sqrt{-a}+a+\frac{1}{3}, & -1<a\le 0\\ a+\frac{1}{3}, & 0<a\end{cases}$ d) $F(a)=\begin{cases}-a-\frac{1}{3}, & a\le -1\\ -\frac{4}{3}a\sqrt{a}+\frac{1}{3}, & -1<a<1\\ a+\frac{1}{3}, & 1\le a\end{cases}$

e) $F(a)=\begin{cases}a+\frac{1}{3}, & a<0\\ a\sqrt{a}+a+\frac{1}{3}, & a\ge 0\end{cases}$ f) $F(a)=\begin{cases}a-\frac{1}{3}, & a\le -1\\ \frac{4}{3}a\sqrt{a}+a, & -1<a<1\\ a-\frac{1}{3}, & 1\le a\end{cases}$

Solution:

The 1st case $a\le 1$ $\left|x^2+a\right|=\begin{cases}x^2+a, & x\in\left(-\infty,-\sqrt{-a}\right)\cup\left(\sqrt{-a},\infty\right)\\ -x^2-a, & x\in\left[-\sqrt{-a},\sqrt{-a}\right]\end{cases}$

$$F(a)=\int_0^1 -\left(x^2+a\right)dx=-\left[\frac{x^3}{3}+ax\right]\Bigg|_0^1=-\frac{1}{3}-a$$

The 2and cas $-1<a\le 0$

$$F(a)=\int_0^{\sqrt{-a}}\left(-x^2-a\right)dx+\int_{\sqrt{-a}}^1\left(x^2+a\right)dx=-\frac{4}{3}a\sqrt{-a}+a+\frac{1}{3}$$

The 3rd case $0<a$

$$F(a)=\int_0^1\left(x^2+a\right)dx=\frac{1}{3}+a$$

The correct answer is c.

AM - XII. 086 Compute the integral: $I=\int_{-1}^{1}\frac{xdx}{\left|\sqrt{1-x}-\sqrt{1+x}\right|}$.

a) $I=1$ b) $I=\frac{2}{3}$ c) $I=0$ d) $I=-1$ e) $I=\frac{\pi}{2}$ f) $I=-\frac{\pi}{2}$

Solution:
We have a integral of a impare function on a symmetrical interval $f\left(x\right)=\dfrac{x}{\left|\sqrt{1-x}-\sqrt{1+x}\right|}$
Hence $I=0$
The correct answer is c.

AM - XII. 114 If $t_1(x)$ and $t_2(x)$ are the solutions of the equation
$t^2+2(x-1)t+4=0$, and
$f(x)=\max\limits_{x\in\mathbf{R}}\{|t_1(x)|,|t_2(x)|\}$, compute the integral $\int\limits_{-2}^{4}f(x)dx$.

a) $13-3\sqrt{5}-2\ln\dfrac{7-3\sqrt{5}}{2}$ b) $13-3\sqrt{5}+2\ln\dfrac{7+3\sqrt{5}}{2}$

c) $13+3\sqrt{5}-2\ln\dfrac{7-3\sqrt{5}}{2}$ d) $13+3\sqrt{5}+2\ln\dfrac{7-3\sqrt{5}}{2}$

e) $13+3\sqrt{5}+2\ln\dfrac{7+3\sqrt{5}}{2}$ f) $13+3\sqrt{5}-2\ln\dfrac{7+3\sqrt{5}}{2}$

Solution:
The equation $t^2+2\left(x-1\right)t+4=0$, has $\Delta=4\left(x^2-2x-3\right)=4\left(x+1\right)\left(x-3\right)$

It $x\in\left(-1,3\right)$, $\Delta<0$ and $t_1,t_2\in\mathrm{C}\setminus\mathrm{R}$ with $\left|t_1\right|=\left|t_2\right|=2$. If $x\in\left(-\infty,-1\right]\cup\left[3,\infty\right)$, $\Delta\ge 0$ and $t_1,t_2\in\mathrm{R}$ with $t_{1,2}=1-x\pm\sqrt{x^2-2x-3}$

$$\left|t_1\left(x\right)\right|=\begin{cases}1-x-\sqrt{x^2-2x-3}, & x\le -1\\ -1+x+\sqrt{x^2-2x-3}, & x\ge 3;\end{cases}\qquad \left|t_2\left(x\right)\right|=\begin{cases}1-x+\sqrt{x^2-2x-3}, & x\le -1\\ -1+x-\sqrt{x^2-2x-3}, & x\ge 3;\end{cases}$$

therefore

$$f\left(x\right)=\begin{cases}1-x+\sqrt{x^2-2x-3}, & x\le -1\\ 2 & x\in\left(-1,3\right)\\ -1+x+\sqrt{x^2-2x-3}, & x\ge 3\end{cases}$$

Separately computing

$$I=\int\sqrt{x^2-2x-3}dx=\int\sqrt{(x-1)^2-4}dx=\frac{1}{2}(x-1)\sqrt{(x-1)^2-4}-2\ln\left|x-1+\sqrt{(x-1)^2-4}\right|$$

Then

$$\int_{-2}^{4}f(x)dx=\int_{-2}^{-1}\left(1-x+\sqrt{x^2-2x-3}\right)dx+\int_{-1}^{3}2dx+\int_{3}^{4}\left(-1+x+\sqrt{x^2-2x-3}\right)dx=$$

$$=13+3\sqrt{5}+2\ln\frac{7-3\sqrt{5}}{2}$$

The correct answer is d.

AM - XII. 133 Compute the integral $I=\int_0^1\frac{2x+1}{x^4+2x^3-x^2-2x+2}dx$.

a) π b) $\frac{\pi}{4}$ c) 0 d) $\frac{\pi}{2}$ e) $\frac{3\pi}{4}$ f) $\frac{3\pi}{2}$

AM - XII. 133

We have: $x^4+2x^3-x^2-2x+2=\left(x^2+x-1\right)^2+1$

Hence $$I=\int_0^1\frac{\left(x^2+x-1\right)'}{1+\left(x^2+x-1\right)^2}dx=arctg\left(x^2+x-1\right)\Big|_0^1=$$

$$=arctg1-arctg(-1)=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$$

The correct answer is d.

AM - XII. 155 Compute the aria A of plane section bounded by the graph of the functions

$$f,g:[-1,1]\to\mathbf{R},\ f(x)=\frac{x^2}{2},\ g(x)=\frac{1}{x^2+1}.$$

a) $A=\frac{\pi}{4}$ b) $A=\frac{\pi}{2}-1$ c) $A=\frac{\pi}{2}-\frac{1}{3}$

d) $A=\frac{\pi}{6}$ e) $A=\frac{\pi}{6}+5$ f) $A=\frac{\pi}{3}+1$

Solution:

$$A=\int_{-1}^{1}\left|f(x)-g(x)\right|dx=\int_{-1}^{1}\left(g(x)-f(x)\right)dx=$$

$$=\int_{-1}^{1}\left(\frac{1}{x^2+1}-\frac{x^2}{2}\right)dx=arctgx\Big|_{-1}^{1}-\frac{x^3}{6}\Big|_{-1}^{1}=\frac{\pi}{2}-\frac{1}{3}$$

The correct answer is c.

AM - XII. 169 Find the volume of the solid body obtained by rotating around the Oy axis of the plane domains between the curves $y=\sqrt{x}$, $y=2$ şi $x=0$.

a) $\frac{\pi}{7}$ b) π c) $\frac{32\pi}{5}$ d) $\frac{21\pi}{2}$ e) $\frac{35\pi}{6}$ f) $\frac{31\pi}{3}$

Solutions:

$$V=\pi\int_{0}^{2}y^4dy=\frac{\pi y^5}{5}\Big|_{0}^{2}=\frac{32\pi}{5}$$

The correct answer is c.

AM - XII. 174 Compute the volume of the body obtained by rotating the subgraph of the function $f:[0,1]\to\mathbf{R}$, $f(x)=\sqrt[4]{x(1-x)}$, around the Ox axis.

a) $\frac{\pi}{2}$ b) $\frac{\pi^2}{8}$ c) $\frac{\pi}{4}$ d) $\frac{\pi^2\sqrt{2}}{2}$ e) 1 f) $\pi^2\sqrt{2}$

Solution:

$$V=\pi\int_0^1\sqrt{x(1-x)}dx; \quad x=\sin^2 t$$

$$V=\pi\int_0^{\frac{\pi}{2}}\sqrt{\sin^2 t\cos^2 t}\,2\sin t\cos t dt=$$

$$=2\pi\int_0^{\frac{\pi}{2}}\sin^2 t\cos^2 t dt=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2 2t dt=$$

$$=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}(1-\cos 4t)\,dt=\frac{\pi^2}{8}$$

The correct answer is b.