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Vol. 64(78), Issue 1 MATHEMATICS–PHISYCS 2019

APPROXIMATE SOLUTIONS BY THE LEAST SQUARES DIFFERENTIAL QUADRATURE METHOD FOR NONLINEAR HEAT TRANSFER PROBLEMS

Mădălina Sofia PAŞCA, Marioara LĂPĂDAT

Abstract

In the present paper we employ a recently introduced approximation method, namely the Least Squares Differential Quadrature Method (LSDQM), in order to compute analytical approximate polynomial solutions for several nonlinear heat transfer problems. [1]

Keywords and phrases: *Nonlinear heat transfer problems, Least Squares Differential Quadrature Method (LSDQM).*

1 Introduction

The heat transfer phenomena are mainly nonlinear and thus they are best modeled by using nonlinear equations. The vast majority of these equations can not be solved analytically using traditional methods and, when the numerical solution of the nonlinear problem is not sufficient, an approximate analytical solution must be computed. In recent years many methods have been developed to compute approximate solutions for nonlinear equations. Among these methods we mention: the Homotopy perturbation method (HPM)([1],[2],[3]), the Homotopy analysis method (HAM)([4],[5]), the Optimal Homotopy Asymptotic Method (OHAM)([6]), the Generalized approximation method (GAM)([7]), the Squared remainder minimization method (SRMM)([8]).
In this paper we compute approximate analytical solutions for some well-known nonlinear heat transfer problems ([15],[16],[17]) modeled by using nonlinear ordinary differential equations.

[1]MSC(2010): 34K28, 45L05

In the next section we will describe the Least Squares Differential Quadrature Method (LSDQM), which allows us to determine analytical approximate polynomial solutions for the above-mentioned type of problems, and in the third section we will compare approximate solutions obtained by using LSDQM with previous approximate solutions presented in ([6],[7],[8],[9]). The computations show that by using LSDQM we obtain approximations with an error smaller than the errors obtained by using other methods.

2 The Least Squares Differential Quadrature Method (LSDQM)

We consider a problem consisting of a nonlinear differential equation of order n:

$$u^{(n)}(t) = F(u^{(n-1)}(t), u^{(n-2)}(t), \cdots, u^{(1)}(t), u(t), t) \tag{1}$$

where F is a continuous function, $t \in [a, b]$ and the boundary conditions:

$$u^{(n-1)}(a) + u^{(n-2)}(a) + \cdots + u^{(1)}(a) + u(a) = \mu_1, \tag{2}$$

$$u^{(n-1)}(b) + u^{(n-2)}(b) + \cdots + u^{(1)}(b) + u(b) = \mu_2. \tag{3}$$

We will consider a numerical meshing of the interval $I = [a, b]$ by means of a partition Δ_M consisting of $M+1$ equidistant points: $a = t_0 < t_1 < t_2 < \cdots < t_{M-1} < t_M = b$. To the equation (1) we attach the following operator:

$$D(u(t)) = u^{(n)}(t) - F(u^{(n-1)}(t), u^{(n-2)}(t), \cdots, u^{(1)}(t), u(t), t). \tag{4}$$

We denote by $\tilde{u}(t)$ an approximate solution of the equation (1). By replacing in D the exact solution $u(t)$ with this approximate solution we obtain the *reminder*:

$$\mathcal{R}(t, \tilde{u}(t)) = D(\tilde{u}(t)), \qquad t \in [a, b]. \tag{5}$$

Definition 1. *We call an* ϵ-***approximate*** *solution of the problem (1 - 3) related to the partition* Δ_M *an approximate polynomial solution which satisfies the following relations:*

$$\mathcal{R}(t_i, \tilde{u}(t_i)) < \epsilon, \qquad i = \overline{0, M}, \tag{6}$$

$$\tilde{u}^{(n-1)}(a) + \tilde{u}^{(n-2)}(a) + \cdots + \tilde{u}^{(1)}(a) + \tilde{u}(a) = \mu_1, \tag{7}$$

$$\tilde{u}^{(n-1)}(b) + \tilde{u}^{(n-2)}(b) + \cdots + \tilde{u}^{(1)}(b) + \tilde{u}(b) = \mu_2. \tag{8}$$

Definition 2. *We consider the sequence of polynomials:*

$$P_N(t) = \sum_{k=0}^{N} c_k t^k, \qquad c_k \in \mathbb{R}, \qquad k = \overline{0, N}. \tag{9}$$

We call the sequence of polynomials $P_N(t)$ ***convergent*** *to the solution of the problem (1 - 3) if:*

$$\lim_{N\to\infty} D(P_N(t)) = 0. \tag{10}$$

We will compute ϵ - approximate polynomial solutions of the type:

$$T_N(t) = \sum_{k=0}^{N} \tilde{c_k} t^k, \tag{11}$$

with the boundary conditions:

$$T_N^{(n-1)}(a) + T_N^{(n-2)}(a) + \cdots + T_N^{(1)}(a) + T_N(a) = \mu_1, \tag{12}$$
$$T_N^{(n-1)}(b) + T_N^{(n-2)}(b) + \cdots + T_N^{(1)}(b) + T_N(b) = \mu_2. \tag{13}$$

The constants $\tilde{c_k}$ are calculated taking the following steps:

- From the boundary conditions we obtain $\tilde{c_0}$ and $\tilde{c_1}$ as functions of $\tilde{c_2}, \tilde{c_3} \cdots \tilde{c_N}$ and replace them in the expression of $T_N(t)$ (which from now on will be a function of $\tilde{c}_2, \tilde{c}_3, \cdots, \tilde{c}_N$ only).

- We attach to the problem (1 - 3) the functional:

$$\mathcal{J}(\tilde{c}_2, \tilde{c}_3, \cdots, \tilde{c}_N) = \sum_{i=0}^{M} \mathcal{R}^2(t_i, T_N(t_i)). \tag{14}$$

- By minimizing the functional (14) we obtain the coefficients $\tilde{c_2}, \tilde{c_3} \cdots \tilde{c_N}$.

- We replace the coefficients $\tilde{c_2}, \tilde{c_3} \cdots \tilde{c_N}$.in the expression (11) and denote by $T_N^0(t) = \sum_{k=0}^{N} \tilde{c_k} t^k$, the analytical approximate polynomial solutions by LSDQM of the problem (1 - 3).

The following convergence theorem is satisfied:

Theorem 1. *The sequence of polynomials $T_N^0(t)$ satisfies the relations:*

$$\lim_{N\to\infty} \mathcal{R}^2(t_i, T_N^0(t_i)) = 0, \qquad i = \overline{0, M}. \tag{15}$$

Proof. Let $u(t)$ be an exact solution of the problem (1 - 3), which means from hypothesis that there exist a sequence of polynomials $P_N(t)$ with $P_N(t) = \sum\limits_{k=0}^{N} c_k t^k$, $c_k \in \mathbb{R}$, $\quad k\overline{0, N}$ converging to $u(t)$: $\quad \lim\limits_{N\to\infty} P_N(t) = u(t), \quad \forall t \in I$.
We know that $\quad \sum\limits_{i=0}^{M} \mathcal{R}^2(t_i, T_N^0(t_i)) \leq \sum\limits_{i=0}^{M} \mathcal{R}^2(t_i, P_N(t_i))$, hence $\lim\limits_{N\to\infty} (\sum\limits_{i=0}^{M} \mathcal{R}^2(t_i, T_N^0(t_i))) \leq \lim\limits_{N\to\infty} (\sum\limits_{i=0}^{M} \mathcal{R}^2(t_i, P_N(t_i)))$.
We conclude that $\quad \lim\limits_{N\to\infty} \mathcal{R}^2(t_i, T_N^0(t_i)) = 0, \qquad i = \overline{0, M}$. $\square$

3 Applications

Application 1

We consider a lumped system of combined convective–radiative heat transfers. The specific heat coefficient is a linear function of temperature ([8],[9],[10]):

$$c = c_a(1 + \gamma(T - T_a))$$

where γ is a constant and c_a is the specific heat at T_a. The cooling process of the system is:

$$\rho V c \frac{dT}{d\tau} + hA(T - T_a) + E\sigma A(T^4 - T_s^4) = 0, \qquad T(0) = T_i$$

Performing the changes of variables: $u = \dfrac{T}{T_i}$, $\;u_a = \dfrac{T_a}{T_i}$, $\;t = \dfrac{\tau(hA)}{\rho V c_a}$, $\;\epsilon_1 = \gamma T_i$, $\epsilon_2 = \dfrac{E\sigma T_i^3}{h}$, $\;u_s = \dfrac{T_s}{T_i}$ and $u_a = u_s = 0$ we obtain the following problem:

$$\frac{du}{dt}(1 + \epsilon_1 u) + u + \epsilon_2 u^4 = 0, \qquad u(0) = 1 \tag{16}$$

Case 1: $\epsilon_1 = 1, \epsilon_2 = 1$

Using the steps outlined in the previous section, we computed the following approximate solutions by LSDQM of the problem (17) :
- third order polynomial: $\tilde{u}(t) = -0.229672t^3 + 0.646583t^2 - 0.968634t + 1$;
- 6th order polynomial: $\tilde{u}(t) = 0.141072t^6 - 0.580724t^5 + 1.0409t^4 - 1.13478t^3 + 0.980631t^2 - 0.99987t + 1$;
- 8th order polynomial: $\tilde{u}(t) = 0.104895t^8 - 0.536881t^7 + 1.21968t^6 - 1.6701t^5 + 1.62301t^4 - 1.29063t^3 + 0.9973t^2 - 0.999999t + 1$

We will compare our approximate solutions with previous solutions: HPM obtained by Ganji et all in ([9]), HAM obtained by Damiary et all in ([11]) and $SRMM$ obtained by Caruntu and Bota in ([8]). Since Eq. (17) does not have a known exact solution, we computed for each approximate solution the relative error as the difference (in absolute value) between the approximate solution and the numerical solution given by the Wolfram Mathematica software.

Table 1 presents the comparison between the solutions obtained by different methods.

Table 1: Comparison of HPM, HAM, SRMM and LSDQ for $\epsilon_1 = \epsilon_2 = 1$

t	HPM	HAM	$SRMM3^{rd}$ deg	$LSDQ3^{rd}$ deg	$LSDQ6^{th}$ deg	$LSDQ8^{th}$ deg
0.1	1.2092	1.866×10^{-2}	1.704×10^{-3}	5.389×10^{-4}	5.073×10^{-5}	4.497×10^{-6}
0.2	7.247×10^{-1}	5.821×10^{-3}	4.66×10^{-4}	1.403×10^{-3}	4.131×10^{-5}	2.084×10^{-6}
0.3	4.276×10^{-1}	6.186×10^{-3}	1.193×10^{-3}	3.378×10^{-3}	8.371×10^{-7}	2.229×10^{-6}
0.4	2.472×10^{-1}	1.709×10^{-2}	2.129×10^{-3}	4.302×10^{-3}	2.565×10^{-6}	3.119×10^{-6}
0.5	1.389×10^{-1}	2.68×10^{-2}	1.995×10^{-3}	3.898×10^{-3}	2.9513×10^{-6}	1.682×10^{-6}
0.6	7.495×10^{-2}	3.528×10^{-2}	9.44×10^{-4}	2.385×10^{-3}	4.653×10^{-5}	9.547×10^{-7}
0.7	3.818×10^{-2}	4.253×10^{-2}	5.48×10^{-4}	3.062×10^{-4}	2.287×10^{-5}	2.216×10^{-6}
0.8	1.78×10^{-2}	4.857×10^{-2}	1.78×10^{-3}	1.578×10^{-3}	1.071×10^{-5}	1.898×10^{-6}
0.9	7.303×10^{-2}	5.345×10^{-2}	1.93×10^{-3}	2.355×10^{-3}	2.980×10^{-6}	4.789×10^{-7}

Case 2: $\epsilon_1 = 1, \epsilon_2 = 0$

Using LSDQM we computed the following second order polynomial approximate solution of equation (17): $\tilde{u}(t) = 0.068967t^2 - 0.501672t + 1$.

As in Case 1, we compare our solution with previous solutions: HPM obtained by Ganji in ([12]), HAM obtained by Abbasbandy in ([13]), $OHAM$ by Marinca and Herisanu in ([6]) and $SRMM$ by Caruntu and Bota in ([8]).

Table 2: Comparison of HPM, HAM, OHAM, SRMM and LSDQ for $\epsilon_1 = 1, \epsilon_2 = 0$

t	HPM	HAM	$OHAM$	$SRMM2^{nd}$ deg	$LSDQ2^{nd}$ deg
0.1	3.350×10^{-2}	3.672×10^{-5}	3.708×10^{-2}	1.561×10^{-3}	1.076×10^{-4}
0.2	4.345×10^{-2}	1.954×10^{-3}	5.366×10^{-2}	8.024×10^{-5}	1.167×10^{-4}
0.3	4.029×10^{-2}	4.091×10^{-3}	5.658×10^{-2}	1.138×10^{-3}	5.698×10^{-5}
0.4	3.071×10^{-2}	5.415×10^{-3}	5.113×10^{-2}	1.669×10^{-3}	4.325×10^{-5}
0.5	1.886×10^{-2}	5.541×10^{-3}	4.118×10^{-2}	1.730×10^{-3}	1.571×10^{-4}
0.6	7.170×10^{-3}	4.432×10^{-3}	2.942×10^{-2}	1.379×10^{-3}	2.600×10^{-4}
0.7	3.062×10^{-3}	2.243×10^{-3}	1.762×10^{-2}	6.736×10^{-4}	3.292×10^{-4}
0.8	1.125×10^{-2}	7.827×10^{-4}	6.855×10^{-3}	3.304×10^{-4}	3.449×10^{-4}
0.9	1.726×10^{-2}	4.372×10^{-3}	2.320×10^{-3}	1.575×10^{-3}	2.903×10^{-4}

Case 3: $\epsilon_1 = 0, \epsilon_2 = 1$

Using the LSDQ, we computed the following approximate solutions:
- 5^{th} order polynomial approximate solution of equation (17):

$$\tilde{u}(t) = -1.37953t^5 + 4.44564t^4 - 5.63484t^3 + 3.84205t^2 - 1.97814t + 1,$$

and - 7^{th} order polynomial approximate solution

$$\tilde{u}(t) = -1.87875t^7 + 8.44452t^6 - 15.8346t^5 + 16.2513t^4 - 10.2726t^3 + 4.58368t^2 - 1.99919t + 1.$$

In Table 3 we present the comparison between the solutions obtained by LSDQM and the solution obtained by Rajabi et all in ([14]) using HPM, Domairry et all in ([11]) using HAM and Caruntu and Bota in ([8]) using $SRMM$.

Application 2

We consider the process of one-dimensional conduction in a slab of thickness L, with the two faces maintained at uniform temperatures T_1 and T_2 with $T_1 > T_2$. The thermal conductivity k is a linear function of temperature ([7], [11], [14]):

$$k = k_2(1 + \mu(T - T_2))$$

Table 3: Comparison of HPM, HAM, SRMM and LSDQ for $\epsilon_1 = 0, \epsilon_2 = 1$

t	HPM	HAM	$SRMM5^{th}$ deg	$LSDQ5^{nd}$ deg	$LSDQ7^{th}$ deg
0.1	2.242×10^{-3}	1.188×10^{-5}	6.081×10^{-5}	2.527×10^{-3}	8.101×10^{-4}
0.2	9.474×10^{-3}	8.720×10^{-4}	1.256×10^{-3}	3.511×10^{-3}	3.789×10^{-4}
0.3	1.799×10^{-2}	1.251×10^{-3}	3.122×10^{-3}	1.400×10^{-3}	5.560×10^{-5}
0.4	2.513×10^{-2}	1.543×10^{-2}	9.676×10^{-3}	7.426×10^{-3}	1.189×10^{-4}
0.5	3.004×10^{-2}	2.093×10^{-2}	1.124×10^{-3}	1.064×10^{-3}	3.735×10^{-4}
0.6	3.270×10^{-2}	3.042×10^{-2}	1.784×10^{-4}	2.875×10^{-3}	3.066×10^{-4}
0.7	3.370×10^{-2}	4.351×10^{-2}	8.140×10^{-4}	1.831×10^{-4}	7.871×10^{-5}
0.8	3.350×10^{-2}	5.957×10^{-2}	7.432×10^{-3}	1.941×10^{-3}	3.225×10^{-5}
0.9	3.230×10^{-2}	7.721×10^{-2}	3.591×10^{-3}	2.691×10^{-3}	1.359×10^{-4}

where μ is a constant and k_2 is the thermal conductivity at T_2 . The problem which describes the process is:

$$\frac{d}{dx}\left(k \cdot \frac{dT}{dx}\right) = 0, \qquad x \in [0, L]$$
$$T(0) = T_1, \qquad T(L) = T_2.$$

Using the dimensionless variables:

$\theta = \dfrac{T - T_2}{T_1 - T_2}, \quad y = \frac{x}{L}, \quad \epsilon = \mu(T_1 - T_2) = \dfrac{k_1 - k_2}{k_2},$ we obtain the following problem:

$$\frac{d^2\theta}{dy^2}(1 + \epsilon\theta) + \epsilon\left(\frac{d\theta}{dy}\right)^2 = 0, \tag{17}$$

$$y \in [0, 1], \quad \theta(0) = 1, \quad \theta(1) = 0. \tag{18}$$

The exact solution of this equation is:

$$\theta_{exact} = \frac{\sqrt{1 - \epsilon(y-1)(\epsilon+2)} - 1}{\epsilon}.$$

Using the LSDQM we computed the following approximate solutions:
- 8^{th} order polynomial approximate solution for the problem (18-19): $\tilde{\theta(y)} = -0.211473y^8 + 0.629685y^7 - 0.787099y^6 + 0.464842y^5 - 0.171033y^4 - 0.0340914y^3 - 0.140741y^2 -$

$0.75009y + 1$, and
- 10^{th} order polynomial approximate solution $\tilde{\theta(y)} = -0.243307y^{10} + 0.979458y^9 - 1.71y^8 + 1.6329y^7 - 0.938876y^6 + 0.302677y^5 - 0.0846321y^4 - 0.0475601y^3 - 0.140633y^2 - 0.750019y + 1$.

In Table 4 we present the comparison between the solutions obtained by LSDQM and the solution obtained by Rajabi et all in ([14]) using HPM, Domairry et all in ([11]) using HAM and GAM and Caruntu and Bota in ([8]) using $SRMM$.

Table 4: Comparison of HPM, HAM,GAM, SRMM and LSDQ for $\epsilon = 1$

t	HPM	HAM	GAM	$SRMM8^{th}$ deg	$LSDQ8^{nd}$ deg	$LSDQ10^{th}$ deg
0.1	1.903×10^{-2}	8.659×10^{-5}	5.406×10^{-6}	5.692×10^{-7}	2.098×10^{-6}	4.440×10^{-7}
0.2	2.790×10^{-2}	9.110×10^{-5}	1.089×10^{-5}	2.598×10^{-6}	2.964×10^{-6}	1.710×10^{-7}
0.3	2.918×10^{-2}	3.066×10^{-4}	1.568×10^{-5}	1.241×10^{-6}	1.614×10^{-6}	2.203×10^{-7}
0.4	2.532×10^{-2}	1.320×10^{-3}	2.005×10^{-5}	4.794×10^{-6}	5.804×10^{-6}	2.737×10^{-7}
0.5	1.863×10^{-2}	3.013×10^{-3}	2.183×10^{-5}	3.750×10^{-7}	1.171×10^{-6}	2.613×10^{-7}
0.6	1.123×10^{-2}	5.239×10^{-3}	2.169×10^{-5}	5.500×10^{-6}	8.370×10^{-6}	2.799×10^{-7}
0.7	4.904×10^{-3}	7.529×10^{-3}	1.887×10^{-5}	4.652×10^{-7}	1.537×10^{-6}	9.978×10^{-7}
0.8	9.110×10^{-4}	8.911×10^{-3}	1.506×10^{-5}	6.420×10^{-6}	5.780×10^{-6}	8.774×10^{-7}
0.9	3.245×10^{-4}	7.550×10^{-3}	8.425×10^{-6}	2.698×10^{-6}	8.202×10^{-6}	1.638×10^{-6}

4 Conclusions

In the present paper we obtain analytical approximate solutions for several nonlinear heat transfer problems using the recently introduced Least Squares Differential Quadrature Method. Using the Least Squares Differential Quadrature Method one obtains the analytical solution of the problem, not only numerical solutions, fact which demonstrates the usefulness of the (LSDQM). The applications presented clearly illustrate the accuracy of the method.

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ANALYTIC APPROXIMATE APPROACHES AND EXACT SOLUTIONS FOR THE EQUATION ON *MHD* FLOW OF A POWER-LAW VISCOUS FLUID

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Abstract

We use a recently modified version of the Optimal Homotopy Asymptotic Method (OHAM) to compute approximate analytic solutions for the equation on magnetohydrodynamic (MHD) flow of a power-law viscous fluid over a stretching sheet and to analyze the effect of the magnetic parameter on the flow. The accuracy of our approximate solutions is emphasized by a comparison with numerical results obtained by using the fourth order Runge-Kutta method. It is shows that the exact solution is computed via OHAM technique. [1]

Keywords and phrases: *optimal homotopy asymptotic method (OHAM), magnetohydrodynamic flow, viscous fluid.*

1 Introduction

The proprieties of viscoelastic materials have been intensively studied in recent years because of their industrial and technological applications such as plastic processing, cosmetics, paint flow, adhesives, accelerators, electrostatic filters, etc [1].

Andersson et al. [2] have further investigated the magnetohydrodynamic flow over a stretching sheet of an electrically conducting incompressible fluid obeying the power-law model. Cortell [8] investigated the laminar boundary layer flow induced in a quiescent visco-elastic fluid by a permeable stretched flat surface with non-linearly (quadratic) velocity. Different methods are applied to study

[1]MSC (2010): 65L60, 76A10, 76D05, 76D10, 76M25, 76M55, 80M25, 34F15

the MHD fluid flow with power-law model, such as: homotopy analysis method (HAM) Abbas et al. [3], Hayat et al. [4], Mabood [10], homotopy perturbation method (HPM) in Raftari et al. [6], [9], homotopy perturbation sumudu transform method (HPSTM) in Sushila et al. [7]. A numerically central-difference scheme is applied in Chen [5].

The aim of the present paper is to propose some accurate, analytic approximate approaches for the equation on magnetohydrodynamic (MHD) flow of a power-law viscous fluid over a stretching sheet and to analyze the effect of the magnetic parameter on the flow using an analytical technique, namely optimal homotopy asymptotic method [11]-[14].

The validity of our procedure, which does not imply the presence of a small parameter in the equation, is based on the construction and determination of the auxiliary functions combined with a convenient way to optimally control the convergence of the solution. The efficiency of the proposed procedure is proves while an accurate solution is explicitly analytically obtained in an iterative way after only one iteration. Two ways to construct some approximate analytic solutions are presented and the exact solution is obtained.

The paper is organized as follows: in the second section the equation on magnetohydrodynamic flow of a power-law viscous fluid over a stretching sheet is presented. In the third section a briefly presentation of the Optimal Homotopy Asymptotic Method, developed in [14] and used in the last part in order to obtain the approximate analytic solutions of the nonlinear differential equation. The last section treats two analytic approaches using rational functions and exponential functions. The exact solution is obtained using exponential functions.

2 Equation of motion

The dimensionless equation on magnetohydrodynamic flow of a power-law viscous fluid over a stretching sheet can be written as [15]:

$$X'''(t) + X(t)X''(t) - (X'(t))^2 - mX'(t) = 0, \tag{1}$$

with the initial/ boundary conditions

$$X(0) = 0,\ X'(0) = 1,\ \lim_{t\to\infty} X'(t) = 0, \tag{2}$$

where $t > 0$, m is magnetic parameter and prime denotes derivative with respect to t.

3 Basic ideas of the optimal homotopy asymptotic method

In [13] the authors compute analytical approximate solutions for equation:

$$L\Big(F(t)\Big) + N\Big(F(t)\Big) = 0, \tag{3}$$

subject to the boundary / initial conditions of the type

$$B\Big(F(t), \frac{dF(t)}{dt}\Big) = 0. \tag{4}$$

For the flow of viscous fluid determined by Eq. (1) with initial/boundary conditions (2), the corresponding operators L, N and B will be introduced in the next section.

We build the following homotopy based on OHAM [11]-[14]:

$$\begin{aligned} &\mathcal{H}\Big[L\Big(F(t,p)\Big),\ H(t,C_i),\ N\Big(F(t,p)\Big)\Big] = \\ &= L\Big(F_0(t)\Big) + p\Big[L\Big(F_1(t,C_i)\Big) - H(t,C_i)N\Big(F_0(t)\Big)\Big] \equiv 0, \end{aligned} \tag{5}$$

where $p \in [0,\ 1]$ is the embedding parameter, L is a linear operator, and

$$H(t, C_i) \neq 0$$

is an auxiliary convergence-control function. H is a function of the variable t and of the parameters C_1, C_2, ..., C_s.

The homotopy (5) satisfies the following properties:

$$\mathcal{H}\Big[L\Big(F(t,0)\Big),\ H(t,C_i),\ N\Big(F(t,0)\Big)\Big] = L\Big(F_0(t)\Big) = 0, \tag{6}$$

$$\mathcal{H}\Big[L\Big(F(t,1)\Big),\ H(t,C_i),\ N\Big(F(t,1)\Big)\Big] = L\Big(F_1(t,C_i)\Big) - H(t,C_i)N\Big(F_0(t)\Big) = 0. \tag{7}$$

Let choose the function F as:

$$F(t,p) = F_0(t) + pF_1(t,C_i), \tag{8}$$

and then from the Eq. (5) we obtain the governing equations of $F_0(t)$ and $F_1(t,C_i)$ by equating the coefficients of p^0 and p^1, respectively:

$$L\Big(F_0(t)\Big) = 0, \quad B\Big(F_0(t), \frac{dF_0(t)}{dt}\Big) = 0, \tag{9}$$

$$L\Big(F_1(t,C_i)\Big) = H(t,C_i)N\Big(F_0(t)\Big),$$
$$B\Big(F_1(t,C_i), \tfrac{dF_1(t,C_i)}{dt}\Big) = 0, \quad i = 1, 2, ..., s. \tag{10}$$

One can find $F_0(t)$ by solving the linear Eq. (9). The nonlinear operator N is defined by:

$$N\left(F_0(t)\right) = \textstyle\sum_{i=1}^{n} h_i(t)g_i(t), \tag{11}$$

so one can compute $F_1(t,C_i)$, where n is a positive integer, and $h_i(t)$ and $g_i(t)$ are known functions that depend on $F_0(t)$ and on N.

Then the general solution of the nonhomogeneous linear equation (10) is obtained by summing the general solution of the corresponding homogeneous equation and a particular solution of the nonhomogeneous equation. But the computation of such a particular solution is not possible in most cases, so the computation of the function $F_1(t,C_i)$, follows the next steps:

- We consider the $F_1(t,C_i)$ of the form:

$$F_1(t,C_i) = \textstyle\sum_{i=1}^{m} H_i(t,h_j(t),C_j)g_i(t), \quad j = 1,\ ...,\ s, \tag{12}$$

or

$$F_1(t,C_i) = \textstyle\sum_{i=1}^{m} H_i(t,g_j(t),C_j)h_i(t), \quad j = 1,\ ...,\ s,$$
$$B\left(F_1(t,C_i),\ \tfrac{dF_1(t,C_i)}{dt}\right) = 0. \tag{13}$$

The above expressions of $H_i(t,h_j(t),C_j)$ contain linear combinations of the functions h_j, $j = 1,\ ...,\ s$ and the parameters C_j, $j = 1,\ ...,\ s$. The summation limit m is an arbitrary positive integer number.

- With Eq. (8), the first-order analytical approximate solution of Eqs. (3) - (4) is:

$$\overline{F}(t,C_i) = F(t,1) = F_0(t) + F_1(t,C_i). \tag{14}$$

- Finally, the convergence-control parameters C_1, C_2, ..., C_s can be optimally computed using methods as: the least square method, the Galerkin method, the collocation method, the Kantorowich method, or the weighted residual method.

With these parameters known, the first-order approximate solution (14) is well-determined.

4 Application of the OHAM to nonlinear problem

Applying our procedure to obtain approximate solutions of Eq. (1) with the initial / boundary conditions Eq. (2).

- *Case 1*

  In the first case of the nonlinear equation Eq. (1), we choose the linear operator of the form:

$$L_f\left(t\right) = f'''(t) + \tfrac{3K}{Kt+1} f''(t), \tag{15}$$

where K is an unknown positive parameter and will be determined later.

Using [13], it is easy to show that the linear operator is not unique.

The initial approximation $f_0(t)$ can be obtained from the following equation:

$$\begin{aligned} & f_0'''(t) + \tfrac{3K}{Kt+1} f_0''(t) = 0, \\ & f_0(0) = 0, \quad f_0'(0) = 1, \quad \lim_{t\to\infty} f_0'(t) = 0, \end{aligned} \tag{16}$$

with solution

$$f_0(t) = \tfrac{1}{K} - \tfrac{1}{K} \cdot \tfrac{1}{Kt+1}. \tag{17}$$

The nonlinear operator $N_f(t)$, corresponding to nonlinear differential Eq. (1), is defined by:

$$N_f\Big(t\Big) = -\tfrac{3K}{Kt+1} f'' + f f'' - (f')^2 - m f'. \tag{18}$$

For the initial approximation $f_0(t)$ given by Eq. (17), the nonlinear operator Eq. (18) becomes:

$$\begin{aligned} & N_{f_0}(t) = -\tfrac{3K}{Kt+1} f_0'' + f_0 f_0'' - (f_0')^2 - m f_0' = \\ & = -\tfrac{m}{(Kt+1)^2} - \tfrac{2}{(Kt+1)^3} + \tfrac{6K^2+1}{(Kt+1)^4}. \end{aligned} \tag{19}$$

Comparing Eqs. (19) and (11), one can write:

$$\begin{aligned} & h_1(t) = -m, \quad g_1(t) = \tfrac{1}{(Kt+1)^2}, \\ & h_2(t) = -2, \quad g_2(t) = \tfrac{1}{(Kt+1)^3}, \\ & h_3(t) = 6K^2 + 1, \quad g_3(t) = \tfrac{1}{(Kt+1)^4}. \end{aligned} \tag{20}$$

The function $f_1(t)$ given by Eq. (12) becomes:

$$f_1(t, C_i) = H_1(t, C_i)\tfrac{1}{(Kt+1)^2} + H_2(t, C_i)\tfrac{1}{(Kt+1)^3} + H_3(t, C_i)\tfrac{1}{(Kt+1)^4}, \tag{21}$$

where we have freedom to choose a lot of possibilities for the unknown functions H_i, $i = 1,\ 2,\ 3$ as follows (see Marinca and Herisanu [13]):

$$\begin{aligned} &H_1(t, C_i) = C_1(Kt+1)^2 + C_2(Kt+1) + C_3, \\ &H_2(t, C_i) = C_4 + \tfrac{C_5}{Kt+1} + \tfrac{C_6}{(Kt+1)^2}, \\ &H_3(t, C_i) = \textstyle\sum_{i=0}^{4} \tfrac{C_{i+7}}{(Kt+1)^{i+2}} + \tfrac{C_{12}t^2}{(Kt+1)^7}. \end{aligned} \tag{22}$$

From Eq. (22) and Eq. (21) we have:

$$f_1(t, C_i) = C_1 + \textstyle\sum_{i=1}^{10} \tfrac{C_{i+1}}{(Kt+1)^i} + \tfrac{C_{12}t^2}{(Kt+1)^{11}}, \tag{23}$$

where

$$C_1 = \sum_{i=3}^{11} (iK - K - 1)C_i, \quad C_2 = -K \cdot \sum_{i=3}^{11} (i-1)C_i.$$

The first-order approximate solution given by Eq. (14) is obtained from Eqs. (17) and (23):

$$\overline{f}(t, C_i) = f_0(t) + f_1(t, C_i) = \tfrac{1}{K} - \tfrac{1}{K} \cdot \tfrac{1}{Kt+1} + C_1 + \textstyle\sum_{i=1}^{10} \tfrac{C_{i+1}}{(Kt+1)^i} + \tfrac{C_{12}t^2}{(Kt+1)^{11}}. \tag{24}$$

- *Case 2*

In the second case of the Eq. (1), we choose the linear operator:

$$L_f\,(t) = f'''(t) - K^2 f'(t), \tag{25}$$

where K is an unknown positive parameter and will be determined later. The initial approximation $f_0(t)$ can be obtained from the following problem:

$$\begin{aligned} &f_0'''(t) - K^2 f_0'(t) = 0, \\ &f_0(0) = 0, \quad f_0'(0) = 1, \quad \lim_{t\to\infty} f_0'(t) = 0, \end{aligned} \tag{26}$$

which has the solution

$$f_0(t) = \tfrac{1}{K}\Big(1 - e^{-Kt}\Big). \tag{27}$$

The nonlinear operator $N_f(t)$, corresponding to nonlinear differential Eq. (1), is defined by:

$$N_f\Big(t\Big) = K^2 f' + f f'' - (f')^2 - m f'. \tag{28}$$

For the initial approximation $f_0(t)$ given by Eq. (27), the nonlinear operator from the Eq. (28) becomes:

$$N_{f_0}(t) = K^2 f_0' + f_0 f_0'' - (f_0')^2 - m f_0' = \Big(K^2 - m - 1\Big) e^{-Kt}. \tag{29}$$

For $K = \sqrt{m+1}$ with $m > -1$, it can be notice that $N_{f_0}(t)$ is identically null, so the **exact solution** is $f_0(t) = \frac{1}{K}\Big(1 - e^{-Kt}\Big)$, as in [2].

If $m \leq -1$, then comparing Eqs. (19) and (11), one can write:

$$h_1(t) = K^2 - m - 1, \quad g_1(t) = e^{-Kt}. \tag{30}$$

The function $f_1(t)$ given by Eq. (12) becomes:

$$f_1(t, C_i) = H_1(t, C_i) e^{-Kt}, \tag{31}$$

where we can choose many possibilities for the unknown functions H_1, as follows (see Marinca and Herisanu [13]):

$$H_1(t, C_i) = C_1 + C_2 t + C_3 t^2 + C_4 e^{-Kt} + C_5 t e^{-Kt} + C_6 t^2 e^{-Kt}. \tag{32}$$

Substituting Eq. (32) into Eq. (31) we have:

$$f_1(t, C_i) = \left(C_1 + C_2 t + C_3 t^2 + C_4 e^{-Kt} + C_5 t e^{-Kt} + C_6 t^2 e^{-Kt}\right) e^{-Kt}, \tag{33}$$

where

$$C_4 = -C_1, \quad C_5 = -C_2 - K C_1.$$

The first-order approximate solution given by Eq. (14) is obtained from Eqs. (17) and (33):

$$\begin{aligned} \overline{f}(t, C_i) &= f_0(t) + f_1(t, C_i) = \tfrac{1}{K}\Big(1 - e^{-Kt}\Big) + \\ &+ \left(C_1 + C_2 t + C_3 t^2 + C_4 e^{-Kt} + C_5 t e^{-Kt} + C_6 t^2 e^{-Kt}\right) e^{-Kt}. \end{aligned} \tag{34}$$

In this way, we can find other solutions as well as in [13].

5 Numerical simulations

In this section, the advance of the OHAM technique is proved using a comparison of our approximate solutions with numerical solution obtained via the fourth-order Runge-Kutta method for different values of the physical parameter m.

The convergence-control parameters K, C_i, $i = \overline{1,\ 12}$ are optimally determined by means of the least-square method using the Mathematica 9.0 software.

Example 1: If $\bar{x}(t)$ is the approximate analytic solution obtained via Optimal Homotopy Asymptotic Method [14], then for $m = -0.75$ the convergence-control parameters are respectively:

$$C_1 = -3.5876955814,\ C_2 = 0.1096257387,\ C_3 = 6.5446309647,$$

$$C_4 = -50.6301828212,\ C_5 = 215.5576321434,\ C_6 = -486.1218585709,$$

$$C_7 = 451.5351597024,\ C_8 = 203.5025800383, C_9 = -921.6008207155,$$

$$C_{10} = 854.9624118554,\ C_{11} = -275.3589466206,\ C_{12} = 2.5850354366,$$

$$K = 0.1778530630.$$

The first-order approximate solutions proposed in [14] becomes:

$$\begin{aligned}\bar{x}_1(t) = 2.0466553957 - \frac{275.3589466206}{(1+0.1778530630t)^{10}} + \frac{854.9624118554}{(1+0.1778530630t)^{9}} - \\ - \frac{921.6008207155}{(1+0.1778530630t)^{8}} + \frac{203.5025800383}{(1+0.1778530630t)^{7}} + \frac{451.5351597024}{(1+0.1778530630t)^{6}} - \\ - \frac{486.1218585709}{(1+0.1778530630t)^{5}} + \frac{215.5576321434}{(1+0.1778530630t)^{4}} - \frac{50.6301828212}{(1+0.1778530630t)^{3}} + \\ + \frac{6.5446309647}{(1+0.1778530630t)^{2}} - \frac{0.4372613717}{1+0.1778530630t} + \frac{2.5850354366t^2}{(1+0.1778530630t)^{11}}\end{aligned} \tag{35}$$

Example 2: For $m = 0.1$ the convergence-control parameters are respectively:

$$C_1 = -5.0017436762,\ C_2 = -0.0879856744,\ C_3 = 0.2947034928,$$

$$C_4 = -3.2746338394,\ C_5 = 19.8793518668,\ C_6 = -69.1185526838,$$

$$C_7 = 134.3757411553,\ C_8 = -123.2607022737,\ C_9 = 0.1131221714,$$

$$C_{10} = 75.4663123965,\ C_{11} = -35.4164117623,\ C_{12} = 0.4347612458,$$

$$K = 0.1679202933.$$

and therefore, the first-order approximate solutions proposed in [14] becomes:

$$\bar{x}_1(t) = 0.9536276893 - \frac{35.4164117623}{(1+0.1679202933t)^{10}} + \frac{75.4663123965}{(1+0.1679202933t)^{9}} +$$
$$+ \frac{0.1131221714}{(1+0.1679202933t)^{8}} - \frac{123.2607022737}{(1+0.1679202933t)^{7}} + \frac{134.3757411553}{(1+0.1679202933t)^{6}} -$$
$$- \frac{69.1185526838}{(1+0.1679202933t)^{5}} + \frac{19.8793518668}{(1+0.1679202933t)^{4}} - \frac{3.2746338394}{(1+0.1679202933t)^{3}} +$$
$$+ \frac{0.2947034928}{(1+0.1679202933t)^{2}} - \frac{0.0125582130}{1+0.1679202933t} + \frac{0.4347612458t^2}{(1+0.1679202933t)^{11}} \tag{36}$$

Example 3: If $m = 0.3$ the convergence-control parameters are respectively:

$$C_1 = -4.3700503876,\ C_2 = -0.5275789347,\ C_3 = -0.1115077163,$$

$$C_4 = -0.0613437484,\ C_5 = 5.3669736027,\ C_6 = -30.8063902687,$$

$$C_7 = 77.8607209154,\ C_8 = -86.6239975041,\ C_9 = 9.6629351317,$$

$$C_{10} = 48.6095463618,\ C_{11} = -24.7875866848,\ C_{12} = 0.4299574779,$$

$$K = 0.1905811550.$$

and therefore:

$$\bar{x}_1(t) = 0.8765292325 - \frac{24.7875866848}{(1+0.1905811550t)^{10}} + \frac{48.6095463618}{(1+0.1905811550t)^9} +$$
$$+ \frac{9.6629351317}{(1+0.1905811550t)^8} - \frac{86.6239975041}{(1+0.1905811550t)^7} + \frac{77.8607209154}{(1+0.1905811550t)^6} -$$
$$- \frac{30.8063902687}{(1+0.1905811550t)^5} + \frac{5.3669736027}{(1+0.1905811550t)^4} - \frac{0.0613437484}{(1+0.1905811550t)^3} -$$
$$- \frac{0.1115077163}{(1+0.1905811550t)^2} + \frac{0.0141206782}{1+0.1905811550t} + \frac{0.4299574779t^2}{(1+0.1905811550t)^{11}} \tag{37}$$

Finally, Tables 1 - 3 and Figures.

1-3 emphasize the accuracy of the OHAM technique by comparing the approximate analytic solutions $\bar{x}_1$, $\bar{x}_1'$ and $\bar{x}_1''$ respectively presented above with the corresponding numerical integration values.

Table 1: The comparison between the approximate solutions $\bar{x}_1$ given by Eq. (37) and the corresponding numerical solutions for $m = 0.3$
(relative errors: $\epsilon_{x_1} = |x_{1_{numerical}} - \bar{x}_1|$)

t	$x_{1_{numerical}}$	$\bar{x}_1$ given by Eq. (37)	ϵ_{x_1}
0	-5.3138 $\cdot 10^{-21}$	2.8421 $\cdot 10^{-14}$	2.8421 $\cdot 10^{-14}$
4/5	0.5247757172	0.5247597553	1.5961 $\cdot 10^{-5}$
8/5	0.7355590457	0.7355718710	1.2825 $\cdot 10^{-5}$
12/5	0.8202230346	0.8202064876	1.6547 $\cdot 10^{-5}$
16/5	0.8542294726	0.8542096875	1.9785 $\cdot 10^{-5}$
4	0.8678886217	0.8678910130	2.3912 $\cdot 10^{-6}$
24/5	0.8733750056	0.8733925671	1.7561 $\cdot 10^{-5}$
28/5	0.8755786866	0.8755948258	1.6139 $\cdot 10^{-5}$
32/5	0.8764638249	0.8764688422	5.0173 $\cdot 10^{-6}$
36/5	0.8768193536	0.8768125402	6.8134 $\cdot 10^{-6}$
8	0.8769621561	0.8769478975	1.4258 $\cdot 10^{-5}$

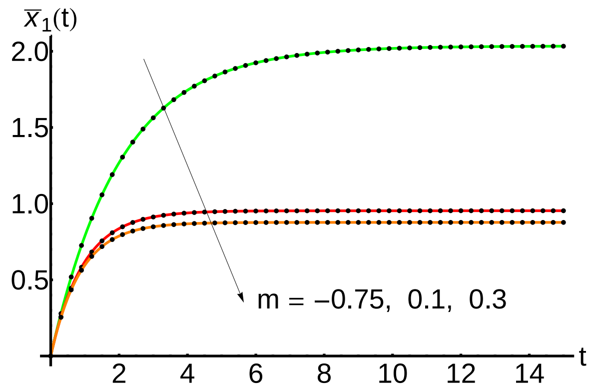

Figure 1: Comparison between the approximate solutions $\bar{x}_1$ given by Eqs. (35), (36), (37) and the corresponding numerical solutions: ——— OHAM solution, ········ numerical solution.

Table 2: The comparison between the approximate solutions $\bar{x}_1'$ from Eq. (37) and the corresponding numerical solutions for $m = 0.3$ (relative errors: $\epsilon_{x_1'} = |x'_{1_{numerical}} - \bar{x}_1'|$)

t	$x'_{1_{numerical}}$	$\bar{x}_1'$ from Eq. (37)	$\epsilon_{x_1'}$
0	1	0.9999999999	$2.7533 \cdot 10^{-14}$
4/5	0.4016636024	0.4017181931	$5.4590 \cdot 10^{-5}$
8/5	0.1613336722	0.1613171078	$1.6564 \cdot 10^{-5}$
12/5	0.0648018043	0.0647710244	$3.0779 \cdot 10^{-5}$
16/5	0.0260285476	0.0260481337	$1.9586 \cdot 10^{-5}$
4	0.0104547212	0.0104828280	$2.8106 \cdot 10^{-5}$
24/5	0.0041992804	0.0042074194	$8.1390 \cdot 10^{-6}$
28/5	0.0016866987	0.0016767703	$9.9284 \cdot 10^{-6}$
32/5	0.0006774861	0.0006615917	$1.5894 \cdot 10^{-5}$
36/5	0.0002721208	0.0002594895	$1.2631 \cdot 10^{-5}$
8	0.0001093012	0.0001035102	$5.7909 \cdot 10^{-6}$

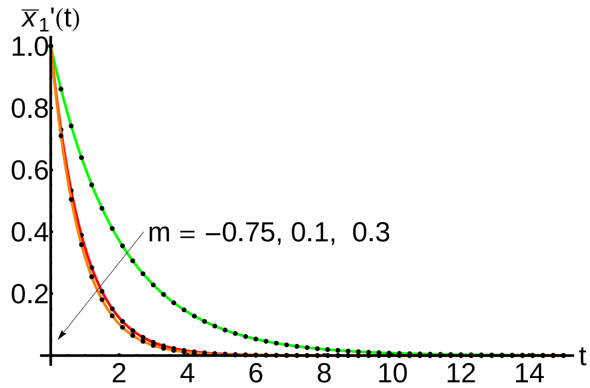

Figure 2: Comparison between the approximate solutions $\bar{x}_1'$ from Eqs. (35), (36), (37) and the corresponding numerical solutions: ——— OHAM solution, ········ numerical solution.

Table 3: The comparison between the approximate solutions $\bar{x}_1''$ from Eq. (37) and the corresponding numerical solutions for $m = 0.3$ (relative errors: $\epsilon_{x_1''} = |x''_{1_{numerical}} - \bar{x}_1''|$)

t	$x''_{1_{numerical}}$	$\bar{x}_1''$ from Eq. (37)	$\epsilon_{x_1''}$
0	-1.1401754250	-1.1401753250	$1.0000 \cdot 10^{-7}$
4/5	-0.4579636131	-0.4578267038	$1.3690 \cdot 10^{-4}$
8/5	-0.1839476837	-0.1840738336	$1.2614 \cdot 10^{-4}$
12/5	-0.0738847299	-0.0738243666	$6.0363 \cdot 10^{-5}$
16/5	-0.0296771408	-0.0296351288	$4.2011 \cdot 10^{-5}$
4	-0.0119201696	-0.0119352076	$1.5037 \cdot 10^{-5}$
24/5	-0.0047879318	-0.0048159808	$2.8049 \cdot 10^{-5}$
28/5	-0.0019231321	-0.0019383749	$1.5242 \cdot 10^{-5}$
32/5	-0.0007724403	-0.0007729498	$5.0954 \cdot 10^{-7}$
36/5	-0.0003102757	-0.0003028727	$7.4029 \cdot 10^{-6}$
8	-0.0001246235	-0.0001157465	$8.8769 \cdot 10^{-6}$

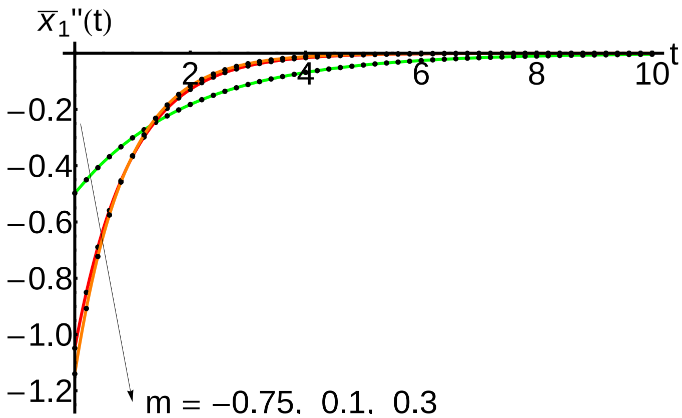

Figure 3: Comparison between the approximate solutions $\bar{x}_1''$ from Eqs. (35), (36), (37) and the corresponding numerical solutions: ——— OHAM solution, ⋯⋯⋯ numerical solution.

Conclusions

We analyze in this paper the equation of the magnetohydrodynamic flow of a power-law viscous fluid over a stretching sheet, from some geometrical point of view. We investigate the stability of the nonlinear differential problem governing this equation. First, we find a Hamilton-Poisson realization, and using specific tools, such as the energy-Casimir method.

Finally, the analytical integration of the nonlinear system (obtained via the Optimal Homotopy Asymptotic Method and presented in [14]) is done and we obtain the exact solution. Numerical integration of the controlled dynamics is obtained via the Optimal Homotopy Asymptotic Method. Numerical simulations and a comparison with Runge-Kutta 4 steps integrator are presented, too.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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PLSM FOR BAGLEY TORVIK EQUATION MODELING THE DEFORMATION RESISTANCE CHARACTERISTICS OF THE POLYMER CONCRETE

Mădălina Sofia PAŞCA, Marioara LĂPĂDAT

Abstract

In this paper we study the Bagley Torvik fractional differential equation which models the deformation resistance characteristics of the polymer concrete. An approximate analytical solution for this equation is obtaining using the Polynomial Least Squares Method (*PLSM*). [1]

Keywords and phrases: *Fractional differential equations, Polynomial Least Square Method(PLSM).*

1 Introduction

Fractional differential equations are widely used in many branches of science such as mathematics, physics, chemistry, biology, and this is the reason why in recent years these equations have been investigated by many authors. In this paper oscillation processes in a viscoelastic medium are described using the fractional Bagley Torvik equation. Using the Polynomial Least Squares Method we obtain an approximate analytical solution of the Bagley Torvik equation, which, compared to the numerical and experimental results for polymer concrete samples presented in ([1]), demonstrates the accuracy of the method. The equation (1) was introduced by Bagley and Torvik in 1983 ([4],[5]) in order to modeling the damping properties of various elastic-plastic materials (polymers, glasses, etc.). In the years that followed, various methods were used to solve numerically or analytically this equation, among which we mention: Homotopy perturbation

[1]MSC (2008): 60H20, 34F15

method ([10]), Discrete spline methods ([6], [8]), Collocation method ([7]), Hybrid functions approximation ([9]).

In engineering, an important role in the development of new materials used in industry, medicine, construction etc is played by polymers. A polymer is a chemical compound with large molecules made of many smaller molecules of the same kind. Some polymers exist naturally and others are produced in laboratories and factories.

In one of the recent paper Temirkhan Aleroev et all ([1]), the Bagley-Torvik equation, presented as:

$$y"(x) + k \cdot D_x^\alpha y(x) + \lambda \cdot y(x) = 0, \tag{1}$$

with $1 < \alpha < 2$, and

$$y(0) = 0, y'(0) = 1, \tag{2}$$

was used in modeling the change in the deformation- strength characteristics of polymer concrete when subjected to loadings. Polymer concrete is represented as a set of granules of mineral extender in an elastic-plastic medium. In this case, the motion of the granule is described by the equation (1), where k is the viscosity modulus of the resin, λ is the rigidity modulus of the resin and α is the elastic-plastic parameter of the medium.

In this paper we obtain an analytical approximate solution for the equation (1) where $D_x^\alpha y(x)$ denotes Caputo's fractional derivative of order α:

$$D_x^\alpha y(x) = \frac{1}{\Gamma(2-\alpha)} \cdot \int_0^x (x-\zeta)^{-(\alpha+1)} \cdot y^{(2)}(\zeta) d\zeta, \qquad 1 < \alpha < 2. \tag{3}$$

In the next section we will introduce the *Polynomial Least Square Method* (*PLSM*) ([2], [3]) which allows us to determine analytical approximate polynomial solutions for fractional ordinary differential equations and in the third section we will compare our approximate solutions with the numerical data presented by Temirkhan Aleroev et all in [1]).

2 The Polynomial Least Squares Method

We denote by $\tilde{y}(x)$ an approximate solution of equation (1). The error obtained by replacing the exact solution $y(x)$ with the approximation $\tilde{y}(x)$ is given by the remainder:

$$\mathcal{R}(\tilde{y}(x)) = \tilde{y}"(x) + k \cdot D_x^\alpha \tilde{y}(x) + \lambda \cdot \tilde{y}(x). \tag{4}$$

For $\epsilon \in \mathbb{R}_+$, we will compute approximate polynomial solutions $\tilde{y}(x)$ of the problem $(1,2)$ on the interval $[0,2]$.

Definition 2.1. *We call an* ϵ-approximate polynomial solution *of the problem* $(1,2)$ *an approximate polynomial solution* $\tilde{y}(x)$ *satisfying the relations*

$$|\mathcal{R}(\tilde{y})| < \epsilon \tag{5}$$

with

$$\tilde{y}(0) = 0. \tag{6}$$

We call a *weak ϵ-approximate polynomial solution* of the problem $(1,2)$ an approximate polynomial solution $\tilde{y}(x)$ satisfying the relation:

$$\int_0^2 |\mathcal{R}(\tilde{y})| dx \leq \epsilon \tag{7}$$

together with the condition (6).

Definition 2.2. *Let* $P_m(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m,\ c_i \in \mathbb{R},\ i = \overline{0,m}$ *be a sequence of polynomials satisfying the condition:*

$$P_m(0) = 0.$$

We call the sequence of polynomials $P_m(x)$ *convergent to the solution of the problem* $(1,2)$ *if* $\lim_{m\to\infty} \mathcal{R}(P_m(x)) = 0$.

We observe that from the hypothesis of the initial problems $(1,2)$ it follows that there exists a sequence of polynomials $P_m(x)$ which converges to the solution of the problem, according to the Weierstrass Theorem on Polynomial Approximation.

We will compute a weak ϵ - approximate polynomial solution, in the sense of the Definition 2.1, of the type:

$$\tilde{y}(x) = \sum_{k=0}^{m} d_k x^k, m > 0 \tag{8}$$

where $d_0, d_1, \cdots, d_m$ are constants which are calculated using the following steps:

- By substituting the approximate solution (8) in the equation (1) we obtain the expression:

$$\mathcal{R}(\tilde{y}) = \tilde{y}"(x) + k \cdot D_x^{\alpha}\tilde{y}(x) + \lambda \cdot \tilde{y}(x). \tag{9}$$

  If we could find $d_0, d_1, \cdots, d_m$ such $\mathcal{R}(\tilde{y}) = 0,\ \tilde{y}(0) = 0$, then by substituting $d_0, d_1, \cdots, d_m$ in (7) we obtain the solutions of equation (1).

- We attach to the problem (1,2) the following functional:

$$\mathcal{J}(d_1, d_2, d_3, \cdots, d_m) = \int_0^2 \mathcal{R}^2(\tilde{y})dx \tag{10}$$

  where d_0 is computed as functions of $d_1, d_2, d_3, \cdots, d_m$ using the initial condition (6).

- We compute the values $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ as the values which give the minimum of the functional $\mathcal{J}$, and the values of d_0 is function of $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ using the initial condition.

- With constants $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ previously determined we consider the polynomial:

$$T_m(x) = \sum_{k=0}^{m} d_k^0 x^k. \tag{11}$$

Theorem 2.1. *The sequence of polynomials $T_m(x)$ from (11) satisfies the property:*

$$\lim_{m\to\infty} \int_0^2 \mathcal{R}^2(T_m(x))dx = 0. \tag{12}$$

Moreover, $\forall \epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ *it follows that* $T_m(x)$ *is a* weak ϵ-approximate polynomial solution *of the problem* $(1, 2)$.

Proof. Based on the way the polynomials $T_m(x)$ are computed and taking into account the relations (9)-(12), the following inequalities are satisfied:

$$0 \le \int_0^2 \mathcal{R}^2(T_m(x))dx \le \int_0^2 \mathcal{R}^2(P_m(x))dx,\ \forall m \in \mathbb{N},$$

where $P_m(x)$ is the sequence of polynomials introduced in Definition 2.2.

It follows that:

$$0 \leq \lim_{m\to\infty} \int_0^2 \mathcal{R}^2(T_m(x))dx \leq \lim_{m\to\infty} \int_0^2 \mathcal{R}^2(P_m(x))dx = 0.$$

and:

$$\lim_{m\to\infty} \int_0^2 \mathcal{R}^2(T_m(x))dx = 0.$$

We obtain that $\forall\epsilon > 0$, $\exists m_o \in \mathbb{N}$, $m > m_0$ it follows that $T_m(x)$ is a *weak ϵ-approximate polynomial solution* of the problem $(1, 2)$. □

In order to find ϵ-approximate polynomial solutions of the problem $(1, 2)$ by using the Polynomial Least Squares Method we will first determine a weak approximate polynomial solutions, $\tilde{y}(x)$. If $|\mathcal{R}(\tilde{y}(x))| < \epsilon$ then $\tilde{y}(x)$ is also an ϵ approximate polynomial solution of the problem.

3 Application

In the equation (1) the remaining parameters are $k = 1.8$ and $\lambda = 93$. These parameter values were obtained during experiments on samples of polymer concrete and the results was presented by Alerov in ([2]).

We consider the following Bagley-Torvik fractional differential equation ([2]):

$$y"(x) + 1.8 \cdot D_x^\alpha y(x) + 93 \cdot y(x) = 0 \tag{13}$$

with $x \in [0, 2]$, $y(0) = 0$ and $y'(0) = 1$.

Using the Polynomial Least Squares Method (PLSM) we follow the steps outlined in the previous section:

- We compute a solution of the type:
  $\tilde{y}(x) = d_0 + d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4 + d_5 \cdot x^5 + d_6 \cdot x^6 + d_7 \cdot x^7 +$
  $+ d_8 \cdot x^8 + d_9 \cdot x^9 + d_{10} \cdot x^{10} + d_{11} \cdot x^{11} + d_{12} \cdot x^{12}$
  and from initial conditions: $\tilde{y}(0) = 0$ and $\tilde{y}'(0) = 1$ we obtain: $d_0 = 0.$ and $d1 = 1$.

- The approximate solution becomes:
  $\tilde{y}(x) = d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4 + d_5 \cdot x^5 + d_6 \cdot x^6 + d_7 \cdot x^7 + d_8 \cdot x^8 +$
  $+ d_9 \cdot x^9 + d_{10} \cdot x^{10} + d_{11} \cdot x^{11} + d_{12} \cdot x^{12}$.

- The corresponding remainder is $\mathcal{R}(\tilde{y}(x))$ (whose expression is too long to be introduced here). Next we compute:

$$\mathcal{J}(d_2, d_3, \cdots, d_9) = \int_0^2 \mathcal{R}^2(\tilde{y}(x))dx$$

  and minimize it obtaining the values:
  $d_2 \to -0.0858228, d_3 \to -11.1994, d_4 \to -3.146, d_5 \to 100.95, d_6 \to -206.366,$ $d_7 \to 184.913, d_8 \to -73.859, d_9 \to 0.148445, d_{10} \to 11.0816, d_{11} \to -3.81496$ and $d_{12} \to 0.425751$

- The approximate analytical solution of the problem (13) using (*PLSM*) is:
  $\tilde{y}(x) = 0.425751x^{12} - 3.81496x^{11} + 11.0816x^{10} + 0.148445x^9 - 73.859x^8 + 184.913x^7 - 206.366x^6 + 100.95x^5 - 3.146x^4 - 11.1994x^3 - 0.0858228x^2 + x.$

Alerov et all in ([1]) presented some experimental results and also a numerical solution for the problem (13).

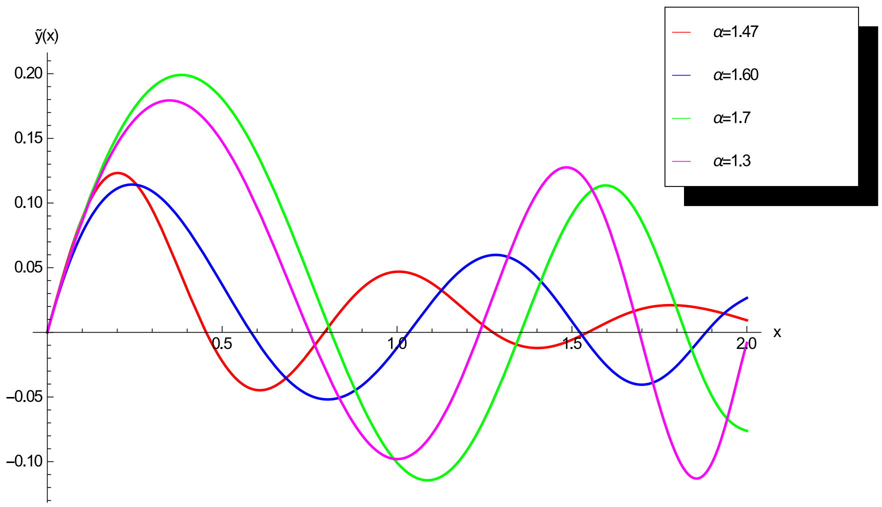
Table 1 present the comparison between experimental results corresponding to the numerical solution proposed by Alerov in ([1]) and our solution obtained using (*PLSM*).

Table 1: Numerical results

x	Experimental results	$PLSM$	$\|Experimental - PLSM\|$
0.25	5×10^{-2}	1.171×10^{-1}	6.716×10^{-2}
0.50	-4×10^{-2}	-2.308×10^{-2}	1.691×10^{-2}
0.75	-1×10^{-2}	-1.559×10^{-2}	5.597×10^{-3}
1.00	2×10^{-2}	4.786×10^{-2}	2.786×10^{-2}
1.25	-1×10^{-2}	3.868×10^{-3}	1.386×10^{-2}
1.50	-1×10^{-2}	$-5..273 \times 10^{-2}$	4.726×10^{-3}

Figure 1 shows the graph of the approximate analytical solution $\tilde{y}(x)$, for different values of α between 1 and 2 (for each approximation a 12 degree polynomial is used).

Figure 1: Graphs of solutions when $1 < \alpha < 2$



4 Conclusions

In the present paper using (PLSM) an analytical solution was obtained for the fractional differential equation which models the deformation-strength characteristics of polymer concrete. The computations performed show that the restults obtained by using (*PLSM*) are in good agreement with the experimental or numerical results obtained by Alerov et all in ([2]). The results were obtained in an easy manner, using minimal time resources, the calculations being made in Wolfram Mathematica.

Additionally, using the Polynomial Least Squares Method one obtains the analytical solution of the problem, not only numerical solutions, fact which demonstrates the usefulness of the (PLSM).

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DYNAMIC ANALYSIS IN AN ECONOMIC MODEL WITH EXPONENTIAL UTILITY FUNCTION AND DELAY

Adina JURATONI, Olivia BUNDĂU

Abstract

In this paper we will analyze a mathematical model associated to an economic growth process with logistic population, exponential utility function and production function of type Cobb-Douglas. Mathematical modeling of this economic growth process leads to an optimal control problem with delay. We show that the model is described by dynamical system of differential functions equations with delay which have the steady state. This steady state exhibits the Hopf bifurcation. [1]

Keywords and phrases: *delay, Hopf bifurcation, mathematical model applied in economies*

1 Introduction

The purpose of this paper is to study the Hopf bifurcation of a economical growth model with logistic population growth and delay between investment and production (time-to-build). We consider a economical growth model with logistic population growth in which production occurs with delay while new capital is installed ([3],[4],[5]). In this economy, the consumer chooses at each moment in time the level of consumption so as to maximize the global utility on the infinite time, given by an exponential function. The mathematical model of this economic growth process leads to an optimal control problem with delay. The optimality conditions, due to the introduction of the time delay, leads to a system of functional differential equations with delay. We determine the steady state of this system and we investigate the local stability of the steady state by analyzing the corresponding transcendental characteristic equation of its linearized system. In the following, by choosing delay as a bifurcation parameter, we show that this model with a delay exhibits the Hopf bifurcation ([6]).

[1]MSC(2010): 49J35, 49K35

2 Setup the model

We consider an economy that is inhabited by infinitely-lived households that, each household has access to a technology that transforms labor and capital into output Y by a neoclassical production function ([2]). We assume that at time t household usees capital goods produced at time $t-\tau$, therefore the output at time t is determined by the following Cobb-Douglas production function

$$Y(t) = K^{\alpha}(t-\tau)L^{1-\alpha}(t),$$

where $Y(t)$ and $K(t-\tau)$ denote aggregate output and aggregate capital stock spent producing goods and $\alpha \in (0,1)$. Considering the aggregate consumption $C(t)$, the capital accumulation equation is

$$\dot{K}(t) = K^{\alpha}(t-\tau)L^{1-\alpha}(t) - C(t) - \delta K(t-\tau),$$

where $\delta \in [0,1]$ is the rate at which capital depreciates. The function L is assumed to evolve according to the logistic law

$$\dot{L}(t) = 2L(t) - L^2(t),$$

the initial population has been normalized to one.

We denote the capital per unit of labor by

$$k = \frac{K}{L}, c(t) = \frac{C(t)}{L(t},$$

for any $L \neq 0$, we can rewrite the capital accumulation equation in intensive form:

$$\dot{k}(t) = k^{\alpha}(t-\tau) - c(t) - (2 - L(t) + \delta)k(t-\tau).$$

In this economy the consumer chooses at each moment in time the level of consumption $c(t)$ such that to maximize the global utility, given by

$$-\frac{1}{\theta}\int_0^{\infty} e^{-\rho t - \theta c(t)} dt,$$

subject to the following constrains

$$\begin{aligned}
\dot{k}(t) &= k^{\alpha}(t-\tau) - c(t) - (2 - L(t) + \delta)k(t-\tau), \\
\dot{L}(t) &= 2L(t) - L^2(t) \\
k(t) &= \varphi(t), t \in [-\tau, 0],
\end{aligned}$$

where $0 < c(t) \leq k^{\alpha}(t-\tau)$, $k(t-\tau)$ is the productive capital at time t, and $\varphi : (-\infty, 0] \to \mathbb{R}_{+}$ is the initial capital function; it needs to be specified in order to identify the relevant history of the state variable. That economical problem, leads us to the following mathematical optimization problem.

Problem P. To determine (c^*, k^*, L^*) which maximizes the following functional

$$-\frac{1}{\theta}\int_{0}^{\infty} e^{-\rho t-\theta c(t)}dt$$

and which verifies

$$\begin{aligned}
\dot{k}(t) &= k^{\alpha}(t-\tau) - c(t) - (2 - L(t) + \delta)k(t-\tau),\\
\dot{L}(t) &= 2L(t) - L^2(t)\\
k(t) &= \varphi(t), t \in [-\tau, 0].
\end{aligned}$$

To solve this optimization problem, we apply the generalized Maximal Principle for time lagged optimal control problems (see Pontryagin et al 1962, [7]). Analogues to [1], the first order conditions of this model are obtained.

Remark 1. The optimal solution of the problem (P) is a solution of the following system of differential equations:

$$\begin{aligned}
\dot{c}(t) &= -\frac{1}{\theta}(\delta + \rho + 2 - L(t) - \alpha k^{\alpha-1}(t-\tau))\\
\dot{k}(t) &= k^{\alpha}(t-\tau) - c(t) - (2 - L(t) + \delta)k(t-\tau),\\
\dot{L}(t) &= 2L(t) - L^2(t)
\end{aligned}$$

3 Local stability analysis and Hopf bifurcation

Generally, the above system is not analytically solvable but we can state some qualitative properties of the solutions. First, we determine the steady states (c^*, k^*, L^*) of the functional differential equation system, which are determined by setting $\dot{c}(t) = \dot{k}(t) = \dot{L}(t) = 0$. From system, results that we have the following

Proposition 3.1. *(Stationary state)* The system of functional differential equation has a unique steady state (c^*, k^*, L^*) which is determined by the following equations:

$$k^* = \left(\frac{\delta+\rho}{\alpha}\right)^{\frac{1}{\alpha-1}},\ c^* = \left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{1}{\alpha-1}},\ L^* = 2.$$

With respect to the transformation

$$x_1(t) = c(t) - c^*,\ x_2(t) = k(t) - k^*,\ x_3(t) = L(t) - L^*$$

the system becomes

$$\begin{cases} \dot{x_1}(t) = -\frac{1}{\theta}[\delta + \rho + 2 - L^* - x_3(t) - \alpha(x_2(t-\tau) + k^*)^{\alpha-1}] \\ \dot{x_2}(t) = -x_1(t) - c^* + (x_2(t-\tau) + k^*)^{\alpha} - (2 - L^* - x_3(t) + \delta)(x_2(t-\tau) + k^*) \\ \dot{x_3}(t) = x_3(t) + L^* - (x_3(t) + L^*)^2 \end{cases}$$

Expanding in Taylor series around $(0,0,0)^t$ and neglect the terms of higher order than the third order, we can rewrite system in the form

$$\begin{cases} \dot{x_1}(t) = a_{010}x_2(t-\tau) + a_{001}x_3(t) + \frac{1}{2}a_{020}x_2^2(t-\tau) + \frac{1}{6}a_{030}x_2^3(t-\tau) + ... \\ \dot{x_2}(t) = b_{100}x_1(t) + b_{010}x_2(t-\tau) + b_{001}x_3(t) + \frac{1}{2}[b_{020}x_2^2(t-\tau) + 2b_{011}x_2(t-\tau)x_3(t)] + \\ \frac{1}{6}b_{030}x_2^3(t-\tau) + ... \\ \dot{x_3}(t) = c_{001}x_3(t) + \frac{1}{2}c_{002}x_3^2(t) + ... \end{cases}$$

where
$a_{010} = \frac{\alpha(\alpha-1)}{\theta}\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}}, a_{020} = \frac{\alpha(\alpha-1)(\alpha-2)}{\theta}\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-3}{\alpha-1}}, a_{001} = \frac{1}{\theta}$,
$a_{030} = \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{\theta}\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-4}{\alpha-1}}, b_{001} = \left(\frac{\delta+\rho}{\alpha}\right)^{\frac{1}{\alpha-1}}$,
$b_{020} = \alpha(\alpha-1)\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}}, b_{030} = \alpha(\alpha-1)(\alpha-2)\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-3}{\alpha-1}}, b_{010} = \rho$,
$b_{011} = 1, b_{100} = -1, c_{001} = -2, c_{002} = -2$.

To investigate the local stability of steady state we linearize the last system. Letting $u(t) = (u_1(t), u_2(t))^t$, be the linearized system variables, linearized system is given by

$$\dot{u}(t) = Au(t) + Bu(t-\tau),$$

where

$$A = \begin{pmatrix} 0 & 0 & a_{001} \\ -1 & 0 & b_{001} \\ 0 & 0 & -2 \end{pmatrix},\quad B = \begin{pmatrix} 0 & a_{010} & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The associated characteristic equation of the linearized system is given by:

$$\lambda^3 + 2\lambda^2 - [\rho\lambda^2 + (\rho - a_{010})\lambda - a_{010}]e^{-\lambda\tau} = 0.$$

Proposition 3.2 If $\tau = 0$ then the characteristic equation is given by

$$\lambda^3 + (2-\rho)\lambda^2 + (\rho - a_{010})\lambda + a_{010} = 0$$

This equation has one positive eigenvalue and two eigenvalues with negative real parts.

Proposition 3.3 Let $\lambda = \lambda(\tau)$ be a solution of characteristic equation. If τ_c, ω_c are given by

$$\omega_c = \frac{\sqrt{2}}{2}\sqrt{\rho^2 + \sqrt{\rho^4 + 4a_{010}^2}},$$

$$\tau_c = \frac{1}{\omega}\arctan\frac{\rho\theta\omega}{\alpha(1-\alpha)\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}}}$$

and $Re\left(\frac{d\lambda}{d\tau}\right)_{\lambda=i\omega,\tau=\tau_c} \neq 0$ then a Hopf bifurcation occurs at the steady state given by (c^*, k^*, L^*) as τ passes through τ_c.

4 Conclusion

In this paper, we analyzed a growth economical model in which the utility is given by an exponential function and Cobb-Douglas production function, with delay for capital and with logistic population growth. Using the delay as a bifurcation parameter we have shown that a Hopf bifurcation occurs when this parameter passes through a critical value .

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A NOTE ON A WELL-POSEDNESS RESULT

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Abstract

We study the well-posedness in the smooth category, for a class of Euler equations. The results obtained by Escher, Bauer and Kolev are extended, in the case $d = 1$, for the class of pseudo-differential operators with x-compactly supported symbols. [1]

Keywords and phrases: *Euler equations, Nash-Moser theorem, elliptic pseudo-differential operators.*

1 Introduction

In two seminal articles V. Arnold [1], respectively D. Ebin, J.E. Marsden [4] created an alternative, to the Nash-Moser schemes, for proving well-posedness results in the smooth category. This alternative is called nowadays the geometric method in hydrodynamics, according to [6]. The main idea is to recast an equation as a geodesic equation, coresponding to a right-invariant metric, on an infinite dimensional manifold, usually a group of diffeomorphisms. The spray equation, via a "no loss, nor gain in spatial regularity" result, will give the possibility of obtaining a Cauchy-Lipschitz type result on Fréchet spaces. In order to prove the smoothness of the spray, on Banach approximations of the tangent bundle, one needs a boundedness result for some multi-linear commutators. In this note we present the boundedness result corresponding to a pseudo-differential operator on $\mathbb{R}$, with an x-compactly supported symbol. This leads to an extension, to inertia operators of pseudo-differential type, of the results presented in [2], in the particular case $d = 1$.

2 The Geometric Approach for Well-posedness

In the sequel we exemplify the geometric method for well-posedness. First of all, let us consider the Lie group $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ defined as:

$$\mathrm{Diff}_{H^\infty}(\mathbb{R}) = \{id + u : \ u \in H^\infty(\mathbb{R}) \text{ and } u' > -1\},$$

[1]MSC (2010): 35Q35, 35S05, 46A61, 58D05

where:

$$H^{\infty}(\mathbb{R}) = \bigcap_{q\geq 0} H^{q}(\mathbb{R}).$$

We will also introduce the Hilbert manifolds:

$$\mathcal{D}^{q}(\mathbb{R}) = \{id + u : \; u \in H^{q}(\mathbb{R}) \text{ and } u' > -1\}.$$

These are actually topological groups for $q > \frac{3}{2}$, but not Lie groups.

We introduce further *the class* $\mathcal{E}^r$ of pseudo-differential operators A as the class of pseudo-differential operators on $\mathbb{R}$, included in Hörmander's class $OpS^{r}_{1,0}$, that are also elliptic, invertible and hermitian, i.e:

$$\sigma_A(x,-\xi) = \overline{\sigma_A(x,\xi)}, \quad x,\xi \in \mathbb{R},$$

where σ_A is the symbol of A. A pseudo-differential operator with a hermitian symbol will send real-valued functions into real-valued functions.

We build inner products on $H^{\infty}(\mathbb{R})$ of the form:

$$\langle u, v\rangle = \int_{\mathbb{R}} Au \cdot v \; dx,$$

where A is a pseudo-differential operator of class $\mathcal{E}^r$. Using right translations,

$$R_{\varphi}(v) := v \circ \varphi,$$

one can obtain a metric on $\mathrm{Diff}_{H^{\infty}}(\mathbb{R})$ via:

$$G_{\varphi}(u_{\varphi}, v_{\varphi}) := \langle R_{\varphi^{-1}}(u_{\varphi}), R_{\varphi^{-1}}(v_{\varphi})\rangle = \int_{\mathbb{R}} A_{\varphi} u_{\varphi} \cdot v_{\varphi} \cdot \varphi' \, dx, \tag{2.1}$$

where $u_{\varphi}, v_{\varphi} \in T_{\varphi}\mathrm{Diff}_{H^{\infty}}(\mathbb{R})$ and A_{φ} is the twisted operator defined as $A_{\varphi} = R_{\varphi} \circ A \circ R_{\varphi}^{-1}$. A geodesic corresponding to the metric G on $\mathrm{Diff}_{H^{\infty}}(\mathbb{R})$ is an extremal curve $\varphi(t)$ of the energy functional:

$$E(\varphi) := \frac{1}{2}\int_0^1 G_{\varphi}(\dot{\varphi}(t), \dot{\varphi}(t))\, dt.$$

If we denote by $u(t) := R_{\varphi^{-1}(t)}\dot{\varphi}(t)$ the *Eulerian velocity* of the geodesic curve $\varphi(t)$, then according to [1] the curve φ(t) is a geodesic if and only if $u(t)$ is a solution of the *Euler-Poincaré equation*:

$$m_t + 2m \cdot u_x + m_x \cdot u = 0, \quad m := Au\,. \tag{2.2}$$

Since A is invertible, the Euler-Poincaré equation (2.2) can be written as

$$u_t = -A^{-1}\left\{2(Au)\cdot u_x + (Au)_x \cdot u\right\}, \tag{2.3}$$

and we call this equation the *Euler–Arnold equation* on $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ for the *inertia operator* A.

On Banach scales $H^q(\mathbb{R})$, the Euler-Arnold equation is of order 1, since if $u \in H^q(\mathbb{R})$ then $A^{-1}\{u \cdot (Au)_x\} \in H^{q-1}(\mathbb{R})$. This derivative loss leads in general to a Nash-Moser approach. In the case investigated in this article the well-posedness in the smooth category can not be investigated with a Nash-Moser scheme, since the space $H^\infty(\mathbb{R})$ is not a tame Fréchet space in the sense of [7]. The lack of suitable interpolation inequalities for the space $C^\infty([-1,1], H^\infty(\mathbb{R}))$ is also a major obstacle in applying a Nash-Moser approach. Thus, in the case presented here, the geometric approach seem to have no contenders.

The geometric approach is based on the following phenomenon: the spray equation (Lagrangian coordinates) behaves better than the Euler-Arnold equation (Eulerian coordinates). One can interpret the spray equation as an ODE on the Hilbert approximations $T\mathcal{D}^q(\mathbb{R})$ of the tangent bundle $T\mathrm{Diff}_{H^\infty}(\mathbb{R})$. We give more details below.

Let φ be the flow of the time dependent vector field u and let $v = \varphi_t$. Then

$$v_t = (u_t + uu_x) \circ \varphi$$

and u solves the Euler equation if and only if (φ, v) is a solution of:

$$\begin{cases} \varphi_t = v \\ v_t = S_\varphi(v) \end{cases} \tag{2.4}$$

where

$$S_\varphi(v) := (R_\varphi \circ S \circ R_{\varphi^{-1}})(v)$$

and

$$S(u) := A^{-1}\{[A,u]u_x - 2(Au)_x\}.$$

It is worth mentioning that if A is a pseudo-differential operator in the class $\mathcal{E}^r$, with $r \geq 1$, then the operator $S(u)$ is of order 0 because the commutator $[A,u]$ is of order less than or equal to $r-1$. That is the essential remark regarding the evolution equation (2.4).

We call the second order vector field F on $\mathrm{Diff}_{H^\infty}(\mathbb{R})$:

$$F : (\varphi, v) \to (\varphi, v, v, S_\varphi(v))$$

the *geodesic spray*, since locally it corresponds to the spray introduced in [9], in the context of Banach manifolds.

The strategy is to prove that the geodesic spray F defined above extends to a smooth mapping on the tangent bundle $T\mathcal{D}^q(\mathbb{R})$. Then one can apply the Cauchy-Lipschitz theorem and there will exist for each $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{R})$ a unique non-extendable solution $(\varphi, v) \in C^\infty(J_q(\varphi_0, v_0), T\mathcal{D}^q(\mathbb{R}))$ of the Cauchy problem (2.4) with $\varphi(0) = \varphi_0$ and $v(0) = v_0$. Here $J_q(\varphi_0, v_0)$ is the maximal interval of existence. The "No loss, nor Gain result" [4] states that $J_{q+1}(\varphi_0, v_0) = J_q(\varphi_0, v_0)$ for an initial data $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{R})$, when $q > r + \frac{1}{2}$. Thus the possibility of having $\bigcap_q J_q(\varphi_0, v_0) = \{0\}$ is excluded and the solution extends to a smooth one:

Theorem 2.1 *Let A be a pseudo-differential operator with an x-compactly symbol, of class $\mathcal{E}^r$ with $r \geq 1$. Then, given any $(\varphi_0, v_0) \in T\mathrm{Diff}_{H^\infty}(\mathbb{R})$, there exists a unique non-extendable geodesic $(\varphi, v) \in C^\infty(J, T\mathrm{Diff}_{H^\infty}(\mathbb{R}))$ on the maximal interval of existence J, which is open and contains 0.*

Corollary 2.2 *The corresponding Euler-Arnold equation* (2.3) *has for any initial data $u_0 \in H^\infty(\mathbb{R})$ a unique non-extendable solution $u \in C^\infty(J, H^\infty(\mathbb{R}))$. The maximal interval of existence J is open and contains 0. Moreover, the solution depends smoothly on u_0.*

In the particular case $A = I - D_x^2$ the Euler-Poincaré equation (2.2) becomes the Camassa-Holm equation:

$$u_t - u_{txx} = 2u_x u_{xx} + u u_{xxx} - 3u u_x \ .$$

For this equation it has been proved in [8] that the solution map can not be more than continuous, for initial data $u_0 \in H^s(\mathbb{R})$. Hence, according to the above corollary, a restriction to smooth initial data $u_0 \in H^\infty(\mathbb{R})$ furnishes smooth dependence, not only continuous.

Following [2] or [5] the major argument used in the proof of Theorem 2.1 relies on proving the smoothness of the twisted operator $\varphi \to A_\varphi$ at the level of $\mathcal{D}^q(\mathbb{R})$. It is worth mentioning that the tangent bundle $T\mathcal{D}^q(\mathbb{R})$ is trivial, hence $T\mathcal{D}^q(\mathbb{R}) \cong \mathcal{D}^q(\mathbb{R}) \times H^q(\mathbb{R})$. The following proposition, together with the "no gain, nor loss" result, will demonstrate the statement of Theorem 2.1, (see [2], for details):

Proposition 2.3 *Let $A \in \mathcal{E}^r$ with an x-compactly supported symbol. Then the mapping:*

$$\varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q \to \mathcal{L}(H^q(\mathbb{R}), H^{q-r}(\mathbb{R}))$$

is smooth for $q > r + 1/2$ and $r \geq 1$.

It has been observed in [2],[5] that the n-th partial Gâteaux derivative of $\varphi \mapsto A_\varphi$, for smooth arguments, is given by:

$$\partial_\varphi^n A_\varphi(v, \delta\varphi_1, \ldots, \delta\varphi_n) = R_\varphi A_n R_\varphi^{-1}(v, \delta\varphi_1, \ldots, \delta\varphi_n),$$

where:

$$A_n := \partial_{\mathrm{id}}^n A_\varphi \in \mathcal{L}^{n+1}(H^\infty(\mathbb{R}), H^\infty(\mathbb{R}))$$

is the $(n+1)$-linear operator defined inductively by $A_0 = A$, and:

$$\begin{aligned} A_{n+1}(u_0, u_1, \ldots, u_{n+1}) &= u_{n+1} D\left(A_n(u_0, u_1, \ldots, u_n)\right) \\ &\quad - \sum_{k=0}^{n} A_n(u_0, u_1, \ldots, u_{n+1} D(u_k), \ldots, u_n), \end{aligned} \tag{2.5}$$

where $D = \frac{d}{dx}$. This observation reduces the problem (see [5, Theorem 3.4]) to showing that each A_n extends to a bounded $(n+1)$-linear operator from $H^q(\mathbb{R})$ to $H^{q-r}(\mathbb{R})$. More precisely:

Lemma 2.4 (Smoothness Lemma) *Let*

$$A : H^\infty(\mathbb{R}) \to H^\infty(\mathbb{R})$$

be a continuous linear operator. Given $q > r + 1/2$ and $r \geq 1$, suppose that A extends to a bounded operator from $H^q(\mathbb{R})$ to $H^{q-r}(\mathbb{R})$. Then:

$$\varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q \to \mathcal{L}(H^q(\mathbb{R}), H^{q-r}(\mathbb{R}))$$

is smooth if and only if each operator A_n defined by (2.5), extends to a bounded $(n+1)$-linear operator in $\mathcal{L}^{n+1}(H^q(\mathbb{R}), H^{q-r}(\mathbb{R}))$.

According to [3, Prop. 6] the operator A_{n+1} can be written in terms of the multilinear commutators:

$$S_{n,P}(u_1, u_2, \ldots, u_n) := [u_1, [u_2 \cdots [u_n, P] \cdots]],$$

for some operators P. Therefore the proof of Proposition 2.3 reduces to proving the following result:

Proposition 2.5 *Let $A = \sigma_A(X, D) \in \mathcal{E}^r$ with an x-compactly suported symbol σ_A, then each A_n extends to a bounded multi-linear operator:*

$$A_n \in \mathcal{L}^{n+1}(H^q(\mathbb{R}), H^{q-r}(\mathbb{R})).$$

Remark 2.6 *Of course, some of the propositions presented above still hold in a more general context, but for our goal we restricted to the class $\mathcal{E}^r$, even when some of the restrictions imposed are redundant.*

3 A Taylor type Estimate

In this section we prove the boundedness result that lies at the root of Proposition 2.3 and implies indirectly the aforementioned well-posedness result:

Proposition 3.1 *Let $P \in OpS^{r+n-1}_{1,0}$ with a hermitian symbol compactly supported in x. For $w \in C_c^\infty(\mathbb{R}), u_1, \ldots, u_n \in C_c^\infty(\mathbb{R})$, we have:*

$$\|S_{n,P}(u_1, \ldots, u_n)w\|_{H^{q-r}} \lesssim \|u_1\|_{H^q} \cdots \|u_n\|_{H^q} \|w\|_{H^{q-1}},$$

where $q > r + 1/2$, $r \geq 1$, and:

$$S_{n,P}(u_1, u_2, \ldots, u_n) := [u_1, [u_2 \cdots [u_n, P] \cdots]].$$

Since $C_c^\infty(\mathbb{R})$ is dense in $H^q(\mathbb{R})$ the above estimate will furnish the boundedness result of A_n, mentioned in Proposition 2.5. In the case $n = 1$, for a single commutator $S_{1,P}(u_1) := [u_1, P]$, the Sobolev boundedness has been proved by M. Taylor in [10], for a general pseudo-differential operator of class $OpS^r_{1,0}$. As far as we know the result can not be extended straightforward to multilinear commutators. Boundedness results like this one use para-differential calculus and seem to be a difficult task, in the case of a general pseudo-differential operator. The restriction to the class of pseudo-differential operators with x-compactly supported symbols, paved us the way to a relative simple proof. We present below the main arguments. The following lemma is standard:

Lemma 3.2 *For a smooth function u and a symbol p from the class $S^r_{1,0}$ the commutator $[u, P]$ has the symbol:*

$$\sigma_{[u,P]}(x, \xi) = \int_{\mathbb{R}} e^{2\pi i \cdot x \cdot \eta} \hat{u}(\eta)[p(x, \xi) - p(x, \xi + \eta)]d\eta,$$

where $\hat{u}$ is the Fourier transform.

Using this lemma one gets a formula for the symbol of $S_{n,P}(u_1, \dots u_n)$:

Proposition 3.3 *Given an operator $P \in OpS^{r+n-1}$, $r \geq 1$, the following formula holds:*

$$\sigma_{S_{n,P}(u_1,\dots u_n)}(x, \xi) =$$

$$\int_{\mathbb{R}} e^{2\pi i x \cdot \eta} \int_{\xi_1+\dots+\xi_n=\eta} \hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\dots\hat{u}_n(\xi_n) \cdot p_n(x, \xi, \xi_1, \dots \xi_n)d\mu d\eta,$$

where μ is the Lebesgue measure on the subspace $\xi_1 + \dots + \xi_n = \eta$ of $\mathbb{R}^{n+1}$ and:

$$p_n(x, \xi, \xi_1, \xi_2, \dots \xi_n) = \sum_{\substack{J \subseteq I_n \\ |J|=k}} (-1)^k p(x, \xi + \sum_{j \in J} \xi_j),$$

for p the symbol of P, with $u_1, \dots u_n \in C_0^\infty(\mathbb{R}, \mathbb{C})$ and $I_n := \{1, 2, \dots, n\}$.

In order to prove the above proposition and to study the boundedness of the multilinear operator S_n a few estimates, on p_n and on its Fourier coefficient $\widehat{p_n}$, are necessary:

Lemma 3.4 *Let $p \in S^{r+n-1}$, $r \geq 1$, an x-compactly supported symbol. The estimate holds:*

$$\|p_n(x, \xi, \xi_1, \dots \xi_n)\| \leq C_{n,r}\langle\xi\rangle^{r-1}\langle\xi_1\rangle^r \dots \langle\xi_n\rangle^r, \tag{3.1}$$

where $C_{n,r}$ depends only on $n \in \mathbb{N}$ and r. Moreover we have the intermediary estimates:

$$\|p_s(x, \xi, \xi_1, \dots \xi_s)\| \leq C_{s,r}\langle\xi\rangle^{r-1+(n-s)}\langle\xi_1\rangle^{r+(n-s)} \dots \langle\xi_s\rangle^{r+(n-s)}, \tag{3.2}$$

for every $s = \overline{1,n}$. Finally for every $t \in \mathbb{N}$ there is a constant $C_n > 0$ such that:

$$|\widehat{p_n}(\eta, \xi, \xi_1 \dots \xi_n)| \leq C_n\langle\eta\rangle^{-t}\langle\xi\rangle^{r-1}\langle\xi_1\rangle^r \dots \langle\xi_n\rangle^r, \tag{3.3}$$

and C_n is independent on $\eta, \xi, \xi_1, \dots \xi_n \in \mathbb{R}$.

Proof 3.5 *In order to obtain the first two estimates we use a similar technique to the one used to Lemma A.6 in [5], because:*

$$\|\partial_\xi^\beta p(x,\xi)\| \leq C_n \langle\xi\rangle^{r-1},$$

for $|\beta| = n$.

The operator corresponding to the symbol $\langle\xi\rangle^2 := 1+\xi^2$ *is* $I - \frac{1}{(2\pi)^2} D_x^2$ *and:*

$$\left(I - \frac{1}{(2\pi)^2} D_x^2\right)^q (e^{-2\pi i\cdot x\cdot\xi}) = \langle\xi\rangle^{2q} \cdot e^{-2\pi i\cdot x\cdot\xi},$$

for every $q \in \mathbb{N}^*$. *Thus:*

$$\begin{aligned}
|\widehat{p_n}(\eta,\xi,\xi_1\dots\xi_n)| &= \left|\int_{\mathbb{R}} e^{-2\pi i\eta\cdot x} p_n(x,\xi,\xi_1,\dots\xi_n)dx\right| \\
&= \langle\eta\rangle^{-2q} \left|\int_{\mathbb{R}} \langle\eta\rangle^{2q} e^{-2\pi i\eta\cdot x} p_n(x,\xi,\xi_1,\dots\xi_n)dx\right| \\
&= \langle\eta\rangle^{-2q} \left|\int_{\mathbb{R}} \left(I - \frac{1}{(2\pi)^2} D_x^2\right)^q (e^{-2\pi i\eta\cdot x}) p_n(x,\xi,\xi_1,\dots\xi_n)dx\right|.
\end{aligned}$$

Since p_n *is* x*-compactly supported, an integration by parts leads to:*

$$\begin{aligned}
|\widehat{p_n}(\eta,\xi,\xi_1\dots\xi_n)| &= \langle\eta\rangle^{-2q} \left|\int_{\mathbb{R}} e^{-2\pi i\eta\cdot x} \left(I - \frac{1}{(2\pi)^2} D_x^2\right)^q p_n(x,\xi,\xi_1,\dots\xi_n)dx\right| \\
&\leq \langle\eta\rangle^{-2q} \int_{\mathbb{R}} \left|\left(I - \frac{1}{(2\pi)^2} D_x^2\right)^q p_n(x,\xi,\xi_1,\dots\xi_n)\right| dx \\
&\leq C_n \langle\eta\rangle^{-2q} \langle\xi\rangle^{r-1} \langle\xi_1\rangle^r \dots \langle\xi_n\rangle^r,
\end{aligned}$$

because the estimate (3.1) *on* p_n *is not affected by a derivative in* x. *For the estimate corresponding to* $t = 2q+1, q \in \mathbb{N}$, *one has to apply a square root to the product obtained from the estimates for* $t = 2q$ *and* $t = 2q+2$.

Proof of Proposition 3.3: For the pseudo-differential operator $[u_2,[u_1,P]]$ applying Lemma 3.2:

$$\sigma_{[u_2,[u_1,P]]}(x,\xi) = \int_{\mathbb{R}} e^{2\pi i x\cdot\xi_2} \hat{u}_2(\xi_2) \cdot \left[\sigma_{[u_1,P]}(x,\xi) - \sigma_{[u_1,P]}(x,\xi+\xi_2)\right] d\xi_2$$

$$= \int_{\mathbb{R}} e^{2\pi i x\cdot\xi_2} \hat{u}_2(\xi_2) \cdot \int_{\mathbb{R}} e^{2\pi i x\cdot\xi_1} \hat{u}_1(\xi_1) \cdot p_2(x,\xi,\xi_1,\xi_2) d\xi_1 d\xi_2$$

for:

$$p_2(x,\xi,\xi_1,\xi_2) := \sum_{\substack{J\subseteq I_2 \\ |J|=k}} (-1)^k p(x,\xi+\sum_{j\in J}\xi_j).$$

We can use the co-area formula in the last expression to obtain:

$$\sigma_{[u_2,[u_1,P]]}(x,\xi) = \int_{\mathbb{R}} e^{2\pi i x\cdot\eta} \cdot \int_{\xi_1+\xi_2=\eta} \hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\cdot p_2(x,\xi,\xi_1,\xi_2)d\mu d\eta$$

The integrand belongs to $\mathcal{L}^1$ by Hölder's inequality, using similar arguments with those of Corollary A.7 in [5]:

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\xi_1+\xi_2=\eta} \left|e^{2\pi i x\cdot\eta}\hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\cdot p_2(x,\xi,\xi_1,\xi_2)\right| d\mu d\eta \\
&\le \int_{\mathbb{R}}\int_{\xi_1+\xi_2=\eta} |\hat{u}_1(\xi_1)|\cdot|\hat{u}_2(\xi_2)|\cdot|p_2(x,\xi,\xi_1,\xi_2)|d\mu d\eta \\
&\le \left(\int_{\mathbb{R}}\langle\tau\rangle^{-2q}d\tau\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}}\langle\tau\rangle^{2q}\left(\int_{\xi_1+\xi_2=\eta}|\hat{u}_1(\xi_1)|\cdot|\hat{u}_2(\xi_2)|\cdot|p_2(x,\xi,\xi_1,\xi_2)|d\mu\right)^2 d\tau\right)^{\frac{1}{2}}
\end{aligned}$$

which is estimated by $\langle\xi\rangle^{r-1+(n-2)}|u_1|_{H^{q+r+(n-2)}}|u_2|_{H^{q+r+(n-2)}}$, for $q > \frac{d}{2}$.

Applied successively, the same idea leads to the symbol formula of $S_{n,P}(u_1,\dots u_n)$.

We are now ready to prove the main result of this section:

Proof of Proposition 3.1: Our previous arguments let us write:

$$\begin{aligned}
&S_{n,P}(u_1,\dots,u_n)(w) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi i x\cdot(\xi_0+\eta)}\int_{\xi_1+\dots+\xi_n=\eta} \hat{w}(\xi_0)\hat{u}_1(\xi_1)\dots\hat{u}_n(\xi_n)p_n(x,\xi_0,\dots,\xi_n)d\mu d\eta d\xi_0 \\
&= \int_{\mathbb{R}} e^{2\pi i x\cdot\eta}\int_{\xi_0+\dots+\xi_n=\eta} \hat{w}(\xi_0)\hat{u}_1(\xi_1)\dots\hat{u}_n(\xi_n)p_n(x,\xi_0,\dots,\xi_n)d\overline{\mu}d\eta\,,
\end{aligned}$$

where $\overline{\mu}$ is the Lebesgue measure on the subspace $\xi_0+\xi_1+\ldots+\xi_n=\eta$ of $\mathbb{R}^{n+2}$. Expressing $p_n(x,\xi_0,\dots,\xi_n)$ via its Fourier transform yields:

$$\begin{aligned}
&S_{n,P}(u_1,\dots,u_n)(w) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi i x\cdot(\eta+\zeta)}\int_{\xi_0+\dots+\xi_n=\eta} \hat{w}(\xi_0)\dots\hat{u}_n(\xi_n)\hat{p}_n(\zeta,\xi_0,\dots,\xi_n)d\overline{\mu}d\eta d\zeta \\
&= \int_{\mathbb{R}} e^{2\pi i x\cdot\zeta}\int_{\mathbb{R}}\int_{\xi_0+\dots+\xi_n=\eta} \hat{w}(\xi_0)\dots\hat{u}_n(\xi_n)\hat{p}_n(\zeta-\eta,\xi_0,\dots,\xi_n)d\overline{\mu}d\eta d\zeta\,.
\end{aligned}$$

Hence, denoting also by $\mathfrak{F}$ the Fourier transform, one gets:

$$\begin{aligned}
&\mathfrak{F}\left(S_{n,P}(u_1,\dots,u_n)(w)\right)(\zeta) \\
&= \int_{\mathbb{R}}\int_{\xi_0+\dots+\xi_n=\eta} \hat{w}(\xi_0)\dots\hat{u}_n(\xi_n)\hat{p}_n(\zeta-\eta,\xi_0,\dots,\xi_n)d\overline{\mu}d\eta\,.
\end{aligned}$$

Next we estimate via Peetre's inequality $\langle\zeta\rangle^{q-r} \lesssim \langle\zeta-\eta\rangle^{q-r}\langle\eta\rangle^{q-r}$ and (3.3):

$$\begin{aligned}&\left|\langle\zeta\rangle^{s-r}\mathfrak{F}\left(S_{n,P}(u_1,\ldots,u_n)(w)\right)(\zeta)\right|\\ &\lesssim \int_{\mathbb{R}}\langle\zeta-\eta\rangle^{q-r-t}\langle\eta\rangle^{s-r}\int_{\xi_0+\cdots+\xi_n=\eta}\langle\xi_0\rangle^{r-1}|\hat{w}(\xi_0)|\prod_{i=1}^{n}\langle\xi_i\rangle^{r}|\hat{u}_i(\xi_i)|d\overline{\mu}d\eta\,,\end{aligned}$$

for some $t\in\mathbb{N}$. For this is a convolution, we can use Young's inequality to estimate:

$$\begin{aligned}&\left\|\langle\zeta\rangle^{q-r}\mathfrak{F}\left(S_{n,P}(u_1,\ldots,u_n)(w)\right)(\zeta)\right\|_{L^2}^2\\ \lesssim&\left(\int_{\mathbb{R}}\langle\zeta-\eta\rangle^{q-r-t}d\eta\right)^2\cdot\int_{\mathbb{R}}\langle\eta\rangle^{2(q-r)}\left(\int_{\xi_0+\cdots+\xi_n=\eta}\langle\xi_0\rangle^{r-1}|\hat{w}(\xi_0)|\prod_{i=1}^{n}\langle\xi_i\rangle^{r}|\hat{u}_i(\xi_i)|d\overline{\mu}\right)^2 d\eta\,.\end{aligned}$$

and the first integral $\int_{\mathbb{R}}\langle\zeta-\eta\rangle^{q-r-t}d\eta$ is bounded since t can be chosen big enough. Let Λ be the operator, such that $\mathfrak{F}\left(\Lambda^r u\right)(\xi)=\langle\xi\rangle^r\mathfrak{F}(u)(\xi)$, which is actually the Fourier multiplier with symbol $\langle\xi\rangle^r$. Let us consider the functions defined by $\tilde{w}(x)=\mathfrak{F}^{-1}(|\hat{w}(\xi_0)|)(x)$ and $\tilde{u}_i(x)=\mathfrak{F}^{-1}(|\hat{u}_i(\xi_i)|)(x)$. First of all, as a consequence of Plancherel's theorem the Sobolev norms of $\tilde{w}$ and w will coincide.

After applying the identity:

$$\mathfrak{F}(f_0\cdot f_1\cdot\ldots f_n)(\eta)=\int_{\xi_0+\cdots+\xi_n=\eta}\hat{f}_0(\xi_0)\cdot\hat{f}_1(\xi_1)\cdot\ldots\cdot\hat{f}_n(\xi_n)d\overline{\mu}$$

to $f_0=\Lambda^{r-1}\tilde{w}$ and $f_i=\Lambda^r\tilde{u}_i,\ i=\overline{1,n}$ we obtain:

$$\int_{\xi_0+\cdots+\xi_n=\eta}\langle\xi_0\rangle^{r-1}|\hat{w}(\xi_0)|\prod_{i=1}^{n}\langle\xi_i\rangle^{r}|\hat{u}_i(\xi_i)|d\overline{\mu}\qquad=\qquad\mathfrak{F}\left(\Lambda^{r-1}\tilde{w}\cdot\prod_{i=1}^{n}\Lambda^r\tilde{u}_i\right)(\eta)\,,$$

Thus:

$$\left\|\langle\zeta\rangle^{q-r}\mathfrak{F}\left(S_{n,P}(u_1,\ldots,u_n)(w)\right)(\zeta)\right\|_{L^2}\lesssim\left\|\Lambda^{r-1}\tilde{w}\cdot\prod_{i=1}^{n}\Lambda^r\tilde{u}_i\right\|_{H^{q-r}}$$

and:

$$\|S_{n,P}(u_1,\ldots,u_n)(w)\|_{H^{q-r}}\lesssim\|w\|_{H^{q-1}}\cdot\prod_{i=1}^{n}\|u_i\|_{H^q}\,,$$

since for $q-r>\frac{1}{2}$ the space $H^{q-r}(\mathbb{R})$ is an algebra.

4 Concluding Remarks

The well-posedness result presented in Corollary 2.2 is also true for $\mathbb{R}^d$ when the Euler-Poincaré equation becomes the EPDiff equation:

$$m_t + \nabla_u m + (\nabla u)^t\, m + (\mathrm{div}\ u)m = 0, \quad m := Au\,, \tag{4.1}$$

with important applications in shape analysis or computational anatomy, see [12]. In order to keep our presentation reasonably short and to avoid some tedious computations we discussed only the case $d = 1$. It is still not very clear if one can drop the restriction $r \geq 1$, for the order of the inertia operator A, since the same restriction occurs with a Nash-Moser approach in the case of the torus S^1.

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MAJORIZATION CRITERIA FOR POLYNOMIAL STABILITY AND INSTABILITY OF EVOLUTION OPERATORS

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Abstract

The aim of this paper is to present majorization criteria for the uniform polynomial stability, respectively for the uniform polynomial instability of evolution operators in Banach spaces. In this sense we establish connections with the exponential case and we give two characterization theorems for the concepts mentioned above. [1]

Keywords and phrases: *majorization criterion, uniform polynomial stability, evolution operator.*

1 Introduction

One of the most important results in the stability theory of evolution operators was obtained by Datko (see [4]) who gave an integral characterization of the uniform exponential stability concept. This paper served as a starting point for many works in which the authors obtained important results concerning the exponential asymptotic behaviors in Banach spaces (see [8, 10, 11, 12]).

Over the last years, the notion of polynomial asymptotic behaviors has been very well developed. It has been considered in the works of Barreira and Valls [2] for evolution operators and Bento and Silva [3] for discrete-time systems. Moreover, important contributions to the study of polynomial stability and instability has been made (see [5, 7, 9]).

[1]MSC (2010): 34D05, 34D20

The present paper will focus on the uniform polynomial stability and uniform polynomial instability of evolution operators. More precisely, we will give majorization criteria for the concepts mentioned above, by extending the techniques used in the exponential case (see [6]) to the polynomial case.

2 Preliminary notions

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|.\|$. The identity operator on X is denoted by I. Also, we consider the sets

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}, \qquad T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$$

Definition 2.1. An application $U : \Delta \to \mathcal{B}(X)$ is said to be an *evolution operator* on X if

(e_1) $U(t,t) = I$ for every $t \geq 0$

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$.

Definition 2.2. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly exponentially stable (u.e.s.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.3. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly polynomially stable (u.p.s.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$(t+1)^\nu \|U(t,s)x\| \leq N(s+1)^\nu \|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.4. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has a uniform polynomial growth (u.p.g.)* if there are $M \geq 1$ and $\omega > 0$ such that

$$(s+1)^\omega \|U(t,s)x\| \leq M(t+1)^\omega \|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Remark 2.5. *It is obvious that*

$$u.p.s. \Rightarrow u.s. \Rightarrow u.p.g.$$

Definition 2.6. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly exponentially instable (u.e.is.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$N\|U(t,s)x\| \geq e^{\nu(t-s)}\|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.7. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly polynomially instable (u.p.is.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$(t+1)^{\nu}\|x\| \leq N(s+1)^{\nu}\|U(t,s)x\|$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.8. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has a uniform polynomial decay (u.p.d.)* if there are $M \geq 1$ and $\omega > 0$ such that

$$(s+1)^{\omega}\|x\| \leq M(t+1)^{\omega}\|U(t,s)x\|$$

for all $(t,s,x) \in \Delta \times X$.

Remark 2.9. *It is easy to see that*

$$u.p.is. \Rightarrow u.is. \Rightarrow u.p.d.$$

Theorem 2.10. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential growth. Then, U is uniformly exponentially stable if and only if there exists a nondecreasing application*

$$\varphi : [1,\infty) \to \mathbb{R}_+ \text{ with } \lim_{t\to\infty} \varphi(t) = \infty \text{ and}$$

$$\varphi(t-s)\|U(t,t_0)x_0\| \leq \|U(s,t_0)x_0\|,$$

for all $(t,s,t_0,x_0) \in T \times X$.

Proof. See [6]. □

Theorem 2.11. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential decay. Then U is uniform exponentially instable if and only if there exists a nondecreasing function $\varphi : [1,\infty) \to \mathbb{R}_+$ with $\lim_{t\to\infty} \varphi(t) = \infty$ and*

$$\varphi(t-s)\|U(s,t_0)x_0\| \leq \|U(t,t_0)x_0\|,$$

for all $(t,s,t_0,x_0) \in T \times X$.

Proof. See [6]. □

3 Main results

Let us consider the evolution operators

$U_1, U_2 : \Delta \to \mathcal{B}(X)$ $U_1(t,s) = U(\ln(t+1), \ln(s+1))$, $U_2(t,s) = U(e^t-1, e^s-1)$,

for all $(t,s) \in \Delta \times X$.

Proposition 3.1. *The evolution operator U_1 is uniformly polynomially stable if and only if U is uniformly exponentially stable.*

Indeed, U u.e.s. is equivalent with

$$\|U_1(t,s)\| = \|U(\ln(t+1), \ln(s+1))\| \le Ne^{-\nu(\ln(t+1)-\ln(s+1))} = N\left(\frac{t+1}{s+1}\right)^{-\nu}$$

which means U_1 u.p.s.

Proposition 3.2. *The evolution operator U is uniformly polynomially stable if and only if U_2 is uniformly exponentially stable.*

Proof. *Necessity* We suppose that U is u.p.s. Then

$$\|U_2(t,s)\| = \|U(e^t-1, e^s-1)\| \le N\left(\frac{e^s}{e^t}\right)^{\nu} = Ne^{-\nu(t-s)},$$

which implies U_2 u.e.s.
Sufficiency Let U_2 be u.e.s. Then

$$\|U_2(t,s)\| = \|U(e^t-1, e^s-1)\| \le Ne^{-\nu(\ln(1+u)-\ln(1+v))} = Ne^{-\nu \ln\frac{1+u}{1+v}} = \\ = Ne^{\ln\left(\frac{1+u}{1+v}\right)^{-\nu}} = N\left(\frac{1+u}{1+v}\right)^{-\nu},$$

which implies U u.p.s. □

Proposition 3.3. *The evolution operator U_1 is uniformly polynomially instable if and only if U is uniformly exponentially instable.*

It results immediately, because U u.e.is. is equivalent with

$$N\|U_1(t,s)\| = N\|U(\ln(t+1), \ln(s+1))x\| \ge e^{\nu(\ln(t+1)-\ln(s+1))}\|x\| = \\ = e^{\nu\ln\frac{t+1}{s+1}}\|x\| = \left(\frac{t+1}{s+1}\right)^{\nu}\|x\|,$$

so U_1 is u.p.is.

Proposition 3.4. *The evolution operator U is uniformly polynomially instable if and only if U_2 is uniformly exponentially instable.*

Proof. If U u.p.is. then

$$N\|U_2(t,s)x\| = N\|U(e^t-1, e^s-1)x\| \geq \left(\frac{e^t}{e^s}\right)^{\nu} \|x\| = e^{\nu(t-s)}\|x\|,$$

which implies that U_2 is u.e.is.
Conversely if U_2 u.e.is., then we have

$$N_2\|U_2(t,s)x\| = N_2\|U(e^t-1, e^s-1)x\| = N_2\|U(u,v)x\| \geq$$
$$\geq e^{\nu(\ln(1+u)-\ln(1+v))}\|x\| = e^{\nu \ln \frac{1+u}{1+v}}\|x\| = \left(\frac{1+u}{1+v}\right)^{\nu} \|x\|,$$

which involves that U is u.p.is. $\square$

The main result of the paper is the next theorem which is an majorization criterion for the concept of uniform polynomial stability of an evolution operator.

Theorem 3.5. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform polynomial growth. Then, U is uniformly polynomially stable if and only if there exists a nondecreasing application*

$$\varphi : [1,\infty) \to \mathbb{R}_+ \text{ with } \lim_{t\to\infty} \varphi(t) = \infty \text{ and}$$

$$\varphi\left(\frac{t+1}{s+1}\right) \|U(t,t_0)x_0\| \leq \|U(s,t_0)x_0\|,$$

for all $(t,s,t_0,x_0) \in T \times X$.

Proof. Necessity.
We suppose that U is u.p.s., which implies using Remark 2.5 and Proposition 3.2 that U has u.p.g. and U_2 is u.e.s., namely U_2 has u.e.g. Then, from the majorization criterion for the uniform exponential stability (see Theorem 2.10) we have that there exists a nondecreasing function $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim\limits_{t\to\infty} \varphi_2(t) = \infty$ and

$$\varphi_2(u-v)\|U_2(u,w)x_0\| \leq \|U_2(v,w)x_0\|, (*)$$

for all $(u, v, w, x_0) \in T \times X$.
Let $u = \ln(t+1),\ v = \ln(s+1),\ w = \ln(t_0+1)$ which implies

$$t = e^u - 1,\ s = e^v - 1,\ t_0 = e^w - 1.$$

Then $\varphi_2(u-v) = \varphi_2\left(\ln\left(\frac{t+1}{s+1}\right)\right) = \varphi\left(\frac{t+1}{s+1}\right)$

$$U(t, t_0) = U(e^u - 1, e^w - 1) = U_2(u, w)$$
$$U(s, t_0) = U(e^v - 1, e^w - 1) = U_2(v, w)$$

If we replace in relation $(*)$ we obtain

$$\varphi\left(\frac{t+1}{s+1}\right)\|U(t, t_0)x_0\| \leq \|U(s, t_0)x_0\|,$$

which is equivalent with the conclusion.
Sufficiency. We suppose that there exists a nondecreasing function

$$\varphi : [1, \infty) \to I\!R_+ \text{ with } \lim_{t\to\infty}\varphi(t) = \infty$$

and

$$\varphi\left(\frac{t+1}{s+1}\right)\|U(t, t_0)x_0\| \leq \|U(s, t_0)x_0\|(**)$$

for all $(t, s, t_0, x_0) \in T \times X$. We have to prove that U is u.p.s, which is equivalent with U_2 is u.e.s.
Let $\varphi_2(x) = \varphi(e^x)$. Let $t = e^u - 1,\ s = e^v - 1,\ t_0 = e^w - 1$, that implies $u = \ln(t+1),\ v = \ln(s+1),\ w = \ln(t_0+1)$. Then

$$\varphi\left(\frac{t+1}{s+1}\right) = \varphi\left(e^{\ln\frac{t+1}{s+1}}\right) = \varphi(e^{\ln(t+1)-\ln(s+1)}) = \varphi(e^{u-v}) = \varphi_2(u-v)$$

$$\|U(t, t_0)x_0\| = \|U(e^u - 1, e^w - 1)x_0\| = \|U_2(u, w)x_0\|$$
$$\|U(s, t_0)x_0\| = \|U(e^v - 1, e^w - 1)x_0\| = \|U_2(v, w)x_0\|$$

If we replace in $(**)$ we obtain

$$\varphi_2(u-v)\|U_2(u, w)x_0\| \leq \|U_2(v, w)x_0\|,$$

so U_2 is u.e.s. $\square$

The following theorem is a majorization criterion for the instability concept.

Theorem 3.6. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform polynomial decay. Then U is uniform polynomially instable if and only if there exists a nondecreasing function*

$$\varphi : [1, \infty) \to \mathbb{R}_+ \text{ with } \lim_{t\to\infty} \varphi(t) = \infty \text{ and}$$

$$\varphi\left(\frac{t+1}{s+1}\right) \|U(s,t_0)x_0\| \le \|U(t,t_0)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$.

Proof. Necessity.
We suppose that U is u.p.is. From Remark 2.9 and Proposition 3.4 we have that U has u.p.d. and U_2 is u.e.is., which implies that U_2 has u.e.d. Then, from the majorization criterion for the uniform exponential instability (see Theorem 2.11) we have that there exists a nondecreasing application $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \varphi_2(t) = \infty$ şi

$$\varphi_2(u-v)\|U_2(v,w)x_0\| \le \|U_2(u,w)x_0\|, (*)$$

for all $(u, v, w, x_0) \in T \times X$.

$$\forall u \in \mathbb{R}_+ \exists t \in \mathbb{R}_+ : u = \ln(t+1) \Rightarrow e^u = t+1 \Rightarrow t = e^u - 1$$

$$\forall v \in \mathbb{R}_+ \exists s \in \mathbb{R}_+ : v = \ln(s+1) \Rightarrow e^v = s+1 \Rightarrow s = e^v - 1$$

$$\forall w \in \mathbb{R}_+ \exists t_0 \in \mathbb{R}_+ : w = \ln(t_0+1) \Rightarrow e^w = t_0+1 \Rightarrow t_0 = e^w - 1$$

Because $u \ge v \ge w$, we obtain $t \ge s \ge t_0$. We compute
$\varphi_2(u-v) = \varphi_2\left(\ln\left(\frac{t+1}{s+1}\right)\right) = \varphi\left(\frac{t+1}{s+1}\right)$, where $\varphi = \varphi_2 \circ \ln$

$$\|U_2(u,w)x_0\| = \|U(e^u-1, e^w-1)x_0\| = \|U(t,t_0)x_0\|$$
$$\|U_2(v,w)x_0\| = \|U(e^v-1, e^w-1)x_0\| = \|U(s,t_0)x_0\|$$

If we replace in $(*)$ we obtain

$$\varphi\left(\frac{t+1}{s+1}\right) \|U(s,t_0)x_0\| \le \|U(t,t_0)x_0\|,$$

so the necessity is proved.

Sufficiency.

We suppose that there exists a nondecreasing function

$$\varphi : [1, \infty) \to \mathbb{R}_+$$

with $\lim_{t\to\infty} \varphi(t) = \infty$ and

$$\varphi\left(\frac{t+1}{s+1}\right) \|U(s,t_0)x_0\| \leq \|U(t,t_0)x_0\| (**)$$

for all $(t, s, t_0, x_0) \in T \times X$. We need to show that U is u.p.is, namely U_2 is u.e.is.

Let $t = e^u - 1,\ s = e^v - 1,\ t_0 = e^w - 1$, which implies

$$u = \ln(t+1),\ v = \ln(s+1),\ w = \ln(t_0+1).$$

Then

$$\varphi\left(\frac{t+1}{s+1}\right) = \varphi\left(e^{\ln\frac{t+1}{s+1}}\right) = \varphi(e^{\ln(t+1)-\ln(s+1)}) = \varphi(e^{u-v}) = \varphi_2(u-v),$$

where $\varphi_2 = \varphi \circ \exp$.

$$\|U(t,t_0)x_0\| = \|U(e^u - 1, e^w - 1)x_0\| = \|U_2(u,w)x_0\|$$

$$\|U(s,t_0)x_0\| = \|U(e^v - 1, e^w - 1)x_0\| = \|U_2(v,w)x_0\|$$

If we replace in the relation $(**)$ we obtain the conclusion. □

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