EDITORIAL BOARD

Tudor BÎNZAR – Editor in Chief

Liviu CĂDARIU
Executive Editor for Mathematics
liviu.cadariu-brailoiu@upt.ro

Duşan POPOV
Executive Editor for Physics
dusan.popov@upt.ro

Camelia ARIEŞANU - Department of Mathematics, Politehnica University Timisoara
Nicolae M. AVRAM - Faculty of Physics, West University of Timisoara
Titu BÂNZARU - Department of Mathematics, Politehnica University Timisoara
Nicolae BOJA - Department of Mathematics, Politehnica University Timisoara
Constantin CORDUNEANU - University of Texas at Arlington, U.S.A.
Emeric DEUTSCH - Politechnic University-Brooklyn, New York, U.S.A.
Sever S. DRAGOMIR - School of Engineering&Science, Victoria University of Melbourne, Australia
Mirela FETEA - Department of Physics, University of Richmond, U.S.A.
Peter G. O. FREUND - Department of Physics and E. Fermi Institute, University of Chicago, U.S.A.
Marian GRECONICI - Depart. of Physical Fundamentals of Engineering, Politehnica University Timisoara
Paşc GĂVRUŢA - Department of Mathematics, Politehnica University Timisoara
Jovo JARIĆ - Faculty of Mathematics, University of Belgrade, Serbia
Maria JIVULESCU - Department of Mathematics, Politehnica University Timisoara
Darko KAPOR - Institute of Physics, University of Novi Sad, Serbia
Octavian LIPOVAN - Department of Mathematics, Politehnica University Timisoara
Dragoljub Lj. MIRJANIĆ - Academy of Science and Art of the Republic of Srpska, Bosnia and Herzegovina
Ioan MUŞCUTARIU - Faculty of Physics, West University of Timisoara
Romeo NEGREA - Department of Mathematics, Politehnica University Timisoara
Emilia PETRIŞOR - Department of Mathematics, Politehnica University Timisoara
Mohsen RAZZAGHI - Dep. of Math. and Statistics, Mississippi State Univ., U.S.A.
Gheorghe ŢIGAN - Department of Mathematics, Politehnica University Timisoara
Ioan ZAHARIE - Dept. of Physical Fundamentals of Engineering, Politehnica University Timisoara

Please consider, when preparing the manuscript, the *Instructions for the Authors* at the end of each issue. Orders, exchange for other journals, manuscripts and all correspondences concerning off prints should be sent to the Executive Editors or to the editorial secretaries at the address:

Politehnica University Timisoara
Department of Mathematics
Victoriei Square, No. 2
300006–Timisoara, Romania
Tel.: +40-256-403099
Fax: +40-(0)256-403109

Politehnica University Timisoara
Department of Physical Fundamentals of Engineering
B-dul. Vasile Parvan, No. 2
300223–Timisoara, Romania
Tel.: +40-256-403391
Fax: +40-(0)256-403392

Contents

SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

DIRECT LIMIT OF MATRIX-RINGS MAY BE UNITAL

Sorin LUGOJAN

Abstract

Although the matrix-rings *M(m;R)* (R is a unital commutative ring) are unital rings, yet their classical direct limit is a non-unital ring. It is presented a direct system of matrix-rings that has a unital ring as a direct limit. [1]

Keywords and phrases: *block-diagonal matrix, direct limit, direct system, matrix-ring.*

1 Introduction

Let $\mathbf{N} = \{1, 2, \ldots\}$, let R be a unital commutative ring of characteristic zero, and let *Rng*, *Ring*, *R-mod* respectively be the classical categories: rings, unital rings and unital morphism, R-modules.

It is known that the direct limit, in *Rng*, of the matrix rings *M(m;R)*, $m \in \mathbf{N}$ is isomorph to $R^{(\mathbf{N}\times\mathbf{N})}$, which belongs to *Rng*, although the matrix-rings belong to *Ring*. By $R^{(\mathbf{N}\times\mathbf{N})}$ we mean the R-module of all mappings $f : \mathbf{N} \times \mathbf{N} \longrightarrow R$ having finite support (only a finite number of images are non-zero). Any such mapping may be considered as a double infinite matrix

$$M = \begin{pmatrix} f(1,1) & f(1,2) & \ldots \\ f(2,1) & f(2,2) & \ldots \\ \vdots & & \end{pmatrix}$$

Thus, one may say that the direct limit of the matrix-ring in *Rng* is the set of infinite matrices having a finite support with the usual operations extended as much as needed. That renders $R^{(\mathbf{N}\times\mathbf{N})}$ as a non-unital ring. We are going to use a special case of block-diagonal matrices:

1. diag(A;r):= diag(A,...,A), where A appears on r slots

[1] MSC (2010): 13A99

2. diag(A,∞):=diag(A,...,A,...)$\in R^{\mathbf{N}\times\mathbf{N}}$

The only possible unit of $R^{\mathbf{N}\times\mathbf{N}}$ is diag(1,∞) , which has no finite support, hence $R^{(\mathbf{N}\times\mathbf{N})} \in$ *Rng*$-$*Ring*.

Is it possible to have a unital ring as the direct limit of matrix-rings ? The answer will be given in the followings. Firstly, one must remark that the direct system of matrix-rings used for the direct limit in *Rng* is not a direct system in *Ring*. Indeed, the mappings of the usual direct system are

$$f_{mn} : M(m;R) \longrightarrow M(n;R)$$

for any $m < n,\ f_{mn}(M) = (\nu_{ij})$, where

$$\nu_{ij} = \begin{cases} \mu_{ij},\ i,j \le n \\ 0,\ else \end{cases}$$

and $M = (\mu_{ij})$. Those are not unital morphisms ($f_{mn}(I_m) \neq I_n$) In fact, the direct limit in *Rng* is the direct limit in the category *R-mod*, plus the remark that the objects and the morphisms implied belong to the category *Rng*, see [1], p. 34. Therefore, if one wants to have a direct limit in *Ring*, one must firstly find a direct system of matrix-rings in *Ring*. But here there is a problem shown in the following theorem.

Theorem 1.1 *If $f : M(m;R) \longrightarrow M(n;R)$ is a* Ring-*morphism for $m < n$, then $m|n$ (m divides n).*

Proof. For the properties of matrices mentioned here, one may see [2] or the Romanian translation [3]. Lets suppose $f : M(m;R) \longrightarrow M(n;R)$ is a *Ring*-morphism. That means:

1. $f(M_1 + M_2) = F(M_1) + f(M_2),\ \forall M_1, M_2 \in M(m;R)$
2. $f(M_1 \cdot M_2) = f(M_1) \cdot f(M_2),\ \forall M_1, M_2 \in M(m;R)$
3. $f(I_m) = I_n$.

The identity matrix I_m may be decomposed into a sum

$$I_m = \sum_{i=1}^{m} E_i$$

where $E_i = (\delta^i_j \cdot \delta^i_k)_{jk}$

is the matrix having just one non-zero entry in the cell
(i, i). The images of E_i, denoted by $F_i = f(E_i),\ i = 1, 2, \ldots .m$, inherit properties of E_i, for example:

a) $\sum_{i=1}^{m} F_i = I_n$

b) the ranks of F_i are all equal.

The a) statement is due to 1) and 3). For b) we consider the matrix M_k, which is obtain from I_m by exchanging the k-th and the (k+*1*)-th rows, that is

$$M_k = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \ldots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The effect of leftwards multiplication of any matrix $M \in M(m; R)$ by M_k is the exchange of the k-th row by the (k+*1*)-th row in M. The effect of rightwards multiplication is the exchange of the k-th by the (k+*1*)-th column in M. All the matrices $M_k,\ k = 1, \ldots, m-1$ are invertible, since their determinants equal
-1, the opposite of 1 in R, which is invertible in R. Then the images by f of M_k, denoted by $N_k = f(M_k)$, are invertible due to 2.) and 3.). By repeated multiplication by M_k, leftwards and rightwards, it is possible to connect any two matrices E_i. Then, any two matrices F_i may be connected by leftwards and rightwards multiplication using invertible matrices in *M(n, R)*, as a consequence of 2.) and 3.). That means F_i has the same rank, $\forall i \in \{1, \ldots, m\}$.

On the other hand for any $B \in M(n; R)$ there is a unique decomposition of B in terms of F_i : $B = \sum_{i=1}^{m} BF_i$, by multiplying the relation a.) by B. Indeed, if $\sum BF_i = \sum CF_i$, it results that $\sum (B - C)F_i = 0$, hence $B - C = 0$. That means BF_i, BF_j have non-zero entries in different cells $\forall i, j$. In order to realise that F_i must have zero-columns. The same is true for F_iB, but here F_i must have zero-rows. Further, all the F_i must have the same number of zero-rows (and zero-columns), else their ranks wouldn't equal. It results that $m|n$.

Corollary 1.2 *The conclusion is that there is no direct system in* Ring *made by the matrix-rings* M(m; R)*, over the index set* ***N****, endowed by the usual order relation.*

Still, we may have a direct system of matrix-rings over **N**, as is stated in the followings.

Lemma 1.3 *The family* M(m; R)*,* $m \in \mathbf{N}$ *is a direct system in* Ring*, over the index set* $\boldsymbol{N}$*, endowed by the relation "divides".*

Proof The set $\mathbf{N}$ and the "divides" relation is a directed set. If $m|n$, there is $r \in \mathbf{N}$ such that $n = r \cdot m$. Then the *Ring*-morphism

$$f_{mn} : M(m;R) \longrightarrow M(n;R),\ f_{mn}(M) = diag(M,r)$$

is the ring-morphism of a direct system in *Ring*. The requirements of direct system are:

1. $f_{mm} = id,\ \forall m \in \mathbf{N}$ (obvious)
2. $f_{np} \circ f_{mn} = f_{mp},\ \forall m|n|p.$

Indeed, supposing that $n = r \cdot m,\ p = s \cdot n$, we have $f_{mn}(M) = diag(M,r)$, then $f_{np}(f_{mn}(M)) = f_{np}(diag(M,r)) = diag(M,rs)$, and also $f_{mp}(M) = diag(M,rs)$.

Theorem 1.4 *The direct limit in* Ring *of the matrix-rings* $M(m;R),\ m \in \mathbf{N}$*, corresponding to the direct system of the Lemma 1.3, is:*

$$L = \{diag(M,\infty)\ |M \in M(m;R), \forall m \in \mathbf{N}\}.$$

Proof. Following [1], p. 33, the direct limit of the direct system in Lemma 1.3 is constructed by considering the direct sum of the R-modules $M(m;R),\ m \in \mathbf{N}$. Those are identified by their images in the direct sum. The direct limit is the quotient set of the direct sum by the R-submodule generated by all the elements $M - f_{mn}(M), \forall M \in M(m;R)$ and $m|n$. That is, the image in the direct sum of any $M \in M(m;R)$ is identified by its image $f_{mn}(M) = diag(M,r)$, where $n = rm$. Hence all the diagonal matrices *diag (M, r)* are identified, and the equivalence class bijectively corresponds to $diag(M,\infty)$. Hence the quotient set is L. L is also a unital ring, and all the implied morphisms are morphisms of unital rings. That results by [1], p. 34 or by straightforward computation.

References

[1] Atiyah, M., MacDonald, I., *Introduction to commutative algebra*, Addison-Wesley Comp. Inc., 1969;

[2] Horn, R.A., Johnson, C.R., *Matrix Analysis*, Cambridge University Press, 1985;

[3] Horn, R.A., Johnson, C.R., *Analiza matricială*, Theta, Bucureşti, 2001.

Sorin Lugojan
Department of Mathematics,
Politehnica University Timisoara,
P-ta Victoriei 2, 300006, Timisoara, Romania
E-mail: sorin.lugojan@upt.ro

SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

SOME NEW REMARKS ON THE FALKNER-SKAN EQUATION: STABILIZATION, INSTABILITY AND LAX FORMULATION

Camelia PETRIŞOR, Remus-Daniel ENE

Abstract

In this paper we study the Falkner-Skan equation. Some stability problems, Lax formulation and an approximate analytic solution by means of the Optimal Homotopy Asymptotic Method (OHAM) were discussed. [1]

Keywords and phrases: *stability, Lax formulation, optimal homotopy asymptotic method (OHAM), nonlinear differential system.*

1 Introduction

The proprieties of viscoelastic materials have been intensively studied in recent years because of their industrial and technological applications such as plastic processing, cosmetics, paint flow, adhesives, accelerators, electrostatic filters, etc [1].

The Falkner-Skan equation describing this proprieties were studied from various points of view: some approximate procedures to solve a boundary layer equations [2], numerical solution [3], existence of a unique smooth solution [4], [5] and [6], was analytically investigated [7] and [8], by using Adomian decomposition method [9] and [10], etc.

The aim of the present paper is to propose a geometrical point of view and an accurate approach to Falkner-Skan equation using an analytical technique, namely optimal homotopy asymptotic method [11], [12], [13].

The validity of our procedure, which does not imply the presence of a small parameter in the equation, is based on the construction and determination of the auxiliary functions combined with a convenient way to optimally control the

[1]MSC (2010): 34-XX, 34A26, 34H05, 34M45, 35A24, 37C10, 49J15, 49K15, 65Lxx, 93C15, 93D05

convergence of the solution. The efficiency of the proposed procedure is proves while an accurate solution is explicitly analytically obtained in an iterative way after only one iteration.

From the geometry point of view, we establish the equilibrium states of the studied system and define a control function. Using specific Hamilton-Poisson geometry methods, namely the energy-Casimir method [14] we are able to study the nonlinear stability of these equilibrium states.

In this paper, a control function is proposed in order to study the stability of the equilibrium states of the system and the numerical integration via the Optimal Homotopy Asymptotic Method of the controlled system is presented.

The paper is organized as follows: in the second paragraph we put the Falkner-Skan equation in a differential system form and find the equilibrium states of the system. In the third section we find a control which preserves the equilibrium states of the system and give a Hamilton-Poisson realization of a controlled system. The fourth section is dedicated to study of stability of the controlled system. In a fifth paragraph is given a Lax formulation for the controlled system and finally in the sixth section a briefly presentation of the Optimal Homotopy Asymptotic Method, developed in [13] and used in the last part in order to obtain the approximate analytic solutions of the controlled system.

2 The Falkner-Skan equation in the flow of a viscous fluid

The dimensionless Falkner-Skan equation in the flow of a viscous fluid can be written as [2], [3], [7], [10]:

$$X'''(t) + X(t)X''(t) + \beta\Big(1 - (X'(t))^2\Big) = 0, \tag{1}$$

with the initial and boundary conditions

$$X(0) = 0,\ X'(0) = 0,\ \lim_{t\to\infty} X'(t) = 1, \tag{2}$$

where $t > 0$, β is a measure of the pressure gradient, and prime denotes derivative with respect to t.

Using the notations:

$$X(t) = x_1(t), \quad X'(t) = x_2(t), \quad X''(t) = x_3(t),$$

the nonlinear equation Eq. (1) becomes:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -\beta\Big(1 - x_2^2\Big) - x_1 x_3 \end{cases}, \qquad t > 0. \tag{3}$$

The nonlinear differential system (3) has an equilibrium state $e^M = (M, 0, 0), M \in \mathbf{R}$ iff $\beta = 0$.

3 The Hamilton-Poisson realization of the system (3)

For the beginning, let us recall very briefly the definitions of general Poisson manifolds and the Hamilton-Poisson systems.

Definition: Let M be a smooth manifold and let $C^\infty(M)$ denote the set of the smooth real functions on M. A **Poisson bracket on** M is a bilinear map from $C^\infty(M) \times C^\infty(M)$ into $C^\infty(M)$, denoted as:

$$(F, G) \mapsto \{F, G\} \in C^\infty(M), F, G \in C^\infty(M)$$

which verifies the following properties:
- skew-symmetry:

$$\{F, G\} = -\{G, F\};$$

- Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0;$$

- Leibniz rule:

$$\{F, G \cdot H\} = \{F, G\} \cdot H + G \cdot \{F, H\}.$$

Proposition: Let $\{\cdot, \cdot\}$ a Poisson structure on $\mathbf{R}^n$. Then for any $f, g \in C^\infty(\mathbf{R}^n, \mathbf{R})$ the following relation holds:

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Let the matrix given by:

$$\Pi = [\{x_i, x_j\}].$$

Proposition: Any Poisson structure $\{\cdot, \cdot\}$ on $\mathbf{R}^n$ is completely determined by the matrix Π via the relation:

$$\{f, g\} = (\bigtriangledown f)^t \Pi (\bigtriangledown g).$$

Definition: A **Hamilton-Poisson system** on $\mathbf{R}^n$ is the triple $(\mathbf{R}^n, \{\cdot,\cdot\}, H)$, where $\{\cdot,\cdot\}$ is a Poisson bracket on $\mathbf{R}^n$ and $H \in C^\infty(\mathbf{R}^n, \mathbf{R})$ is the energy (Hamiltonian). Its dynamics is described by the following differential equations system:

$$\dot{x} = \Pi \cdot \nabla H$$

where $x = (x_1, x_2, ...x_n)^t$.

Definition: Let $\{\cdot,\cdot\}$ a Poisson structure on $\mathbf{R}^n$. A **Casimir** of the configuration $(\mathbf{R}^n, \{\cdot,\cdot\})$ is a smooth function $C \in C^\infty(\mathbf{R}^n, \mathbf{R})$ which satisfy:

$$\{f, C\} = 0, \forall f \in C^\infty(\mathbf{R}^n, \mathbf{R}).$$

Let us employ the control $u \in C^\infty(\mathbf{R}^3, \mathbf{R})$,

$$u(x_1, x_2, x_3) = (0\ ,\ x_1 x_2\ ,\ -x_2^2 - x_1^2 x_2), \tag{4}$$

for the system (3). The controlled system (3)−(4), explicitly given by:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 + x_1 x_2 \qquad , \\ x_3' = -x_1 x_3 - x_2^2 - x_1^2 x_2. \end{cases} \qquad t > 0, \tag{5}$$

Proposition: The controlled system (5) has the Hamilton-Poisson realization

$$(\mathbf{R}^3, \Pi_-, H),$$

where

$$\Pi_- = \begin{bmatrix} 0 & 1 & -x_1 \\ -1 & 0 & x_2 \\ x_1 & -x_2 & 0 \end{bmatrix}$$

is the minus Lie-Poisson structure and

$$H(x_1, x_2, x_3) = \frac{1}{2} x_2^2 - x_1 x_3 - x_1^2 x_2$$

is the Hamiltonian.

Proof: Indeed, we have:

$$\Pi_- \cdot \nabla H = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

and the matrix Π_- is a Poisson matrix, see [15].

The next step is to find the Casimirs of the configuration described by the above Proposition. Since the Poisson structure is degenerate, there exist Casimir functions. The defining equations for the Casimir functions, denoted by C, are

$$\Pi^{ij}\partial_j C = 0.$$

It is easy to see that there exists only one functionally independent Casimir of our Poisson configuration, given by $C : \mathbf{R}^3 \to \mathbf{R}$,

$$C(x_1, x_2, x_3) = -x_3 - x_1 x_2.$$

Consequently, the phase curves of the dynamics Eq. (5) are the intersections of the surfaces $H(x_1, x_2, x_3) = const.$ and $C(x_1, x_2, x_3) = const.$.

4 Stability Problem

The concept of stability is an important issue for any differential equation. The nonlinear stability of the equilibrium point of a dynamical system can be studied using the tools of mechanical geometry, so this is another good reason to find a Hamilton -Poisson realization. For more details, see [15]. We start this section with a short review of the most important notions.

Definition: An equilibrium state x_e is said to be **nonlinear stable** if for each neighbourhood U of x_e in D there is a neighbourhood V of x_e in U such that trajectory $x(t)$ initially in V never leaves U.

This definition supposes well-defined dynamics and a specified topology. In terms of a norm $\| \|$, nonlinear stability means that for each $\varepsilon > 0$ there is $\delta > 0$ such that if

$$\|x(0) - x_e\| < \delta$$

then

$$\|x(t) - x_e\| < \varepsilon, \quad (\forall) \quad t > 0.$$

It is clear that nonlinear stability implies spectral stability; the converse is not always true.

The equilibrium states of the dynamics Eq. (1) are

$$e^M = (M, 0, 0), \quad M \in \mathbf{R}.$$

Proposition 1: For the equilibrium states $e^M = (M, 0, 0)$ the following statements hold:

a) $e^M = (M, 0, 0)$ are unstable for $M > 0$;
b) $e^M = (M, 0, 0)$ are unstable for $M = 0$

Proof: We will use energy-Casimir method, see [15] for details. Let

$$F_\varphi(x_1, x_2, x_3) = H(x_1, x_2, x_3) + \varphi[C(x_1, x_2, x_3)] =$$

$$= \frac{1}{2}x_2^2 - x_1x_3 - x_1^2x_2 + \varphi(-x_3 - x_1x_2)$$

be the energy-Casimir function, where $\varphi : \mathbf{R} \to \mathbf{R}$ is a smooth real valued function.

Now, the first variation of F_φ is given by

$$\delta F_\varphi(x_1, x_2, x_3) = x_2\delta x_2 - x_1\delta x_3 - x_3\delta x_1 - 2x_1x_2\delta x_1 - x_1^2\delta x_2 +$$

$$+\dot{\varphi}\,(-x_3 - x_1x_2) \cdot (-x_1\delta x_2 - x_2\delta x_1 - \delta x_3)$$

so we obtain

$$\delta F_\varphi(e^M) = [M + \dot{\varphi}\,(0)] \cdot (-M\delta x_2 - \delta x_3)$$

that is equals zero for any $M \in \mathbf{R}^*$ if and only if

$$\dot{\varphi}\,(0) = -M. \tag{6}$$

The second variation of F_φ at the equilibrium of interest is given by

$$\delta^2 F_\varphi(e^M) = [\ddot{\varphi}(0)]^{-1} \cdot [\ddot{\varphi}(0)\delta x_3 - \delta x_1 + M \cdot \ddot{\varphi}(0)\delta x_2]^2 +$$

$$+[\ddot{\varphi}(0)]^{-1}[1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2]^{-1} \cdot \Big[\Big(1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2\Big)\delta x_2 +$$

$$+\Big(M\ddot{\varphi}(0) - M\Big)\delta x_1\Big]^2 + [1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2]^{-1} \cdot \big[-1 + M^2\ddot{\varphi}(0) - M^2\big]\,\Big(\delta x_1\Big)^2.$$

If we choose now φ such that the relation (6) is valid and $\delta^2 F_\varphi(e^M)$ is positive defined, i.e.

$$\ddot{\varphi}(0) > 0 \quad \text{and} \quad 1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2 > 0 \quad \text{and} \quad -1 + M^2\ddot{\varphi}(0) - M^2 > 0$$

then the second variation of F_φ at the equilibrium of interest is positive defined.

We can assume that $M > 0$. From these inequalities we deduce that:

$$\frac{M^2 + 1}{M^2} < \varphi''(0) < \frac{M^2 + M\sqrt{M^2 + 4}}{2M^2},$$

that implies

$$2M^2+2<M^2+M\sqrt{M^2+4} \quad\Rightarrow\quad (M^2+2)^2<M^2(M^2+4) \quad\Rightarrow\quad 4<0,$$

that is false.

Therefore, the equilibrium state $e^M(M,0,0)$ is unstable.

In the same way, we conclude that $e^M(M,0,0)$ is unstable for $M=0$. □

Table 1: The comparison between the approximate solutions $\bar{x}_1$ given by Eq. (7) and the corresponding numerical solutions for $\beta=0$
(relative errors: $\epsilon_{x_1}=|x_{1_{numerical}}-\bar{x}_1|$)

t	$x_{1_{numerical}}$	$\bar{x}_1$ given by Eq. (7)	ϵ_{x_1}
0	$1.5671 \cdot 10^{-25}$	$-1.3322 \cdot 10^{-15}$	$1.3322 \cdot 10^{-15}$
4/5	0.149674539444	0.149401535388	$2.73004 \cdot 10^{-4}$
8/5	0.582956328320	0.582978942361	$2.2614 \cdot 10^{-5}$
12/5	1.231527648000	1.231539489179	$1.1841 \cdot 10^{-5}$
16/5	1.990581010375	1.990607740256	$2.6729 \cdot 10^{-5}$
4	2.783886492275	2.783817337929	$6.9154 \cdot 10^{-5}$
24/5	3.583254092715	3.583303973631	$4.9880 \cdot 10^{-5}$
28/5	4.383220411026	4.383289581820	$6.9170 \cdot 10^{-5}$
32/5	5.183219409763	5.183234915896	$1.5506 \cdot 10^{-5}$
36/5	5.983219388168	5.983199995268	$1.9392 \cdot 10^{-5}$
8	6.783219382599	6.783194882759	$2.4499 \cdot 10^{-5}$

5 Lax formulation

Let introduce the matrices:

$$L=\begin{pmatrix} \frac{1}{2}x_2^2 & -x_1x_3-x_3-\frac{1}{8}x_2^4-\frac{1}{2}(-x_1x_3-x_1^2x_2)^2 \\ 1 & -x_1x_3-x_1^2x_2 \end{pmatrix} \quad ,$$

$$B=\begin{pmatrix} 1 & -x_2(x_3+x_1x_2) \\ 0 & 1 \end{pmatrix}.$$

Then an easy computation we can establish the following result:

Theorem 5.1 *The controlled system (5) have a Lax formulation, i.e., it can be put in the equivalent form:*

$$\frac{dL}{dt}=\Big[L\ ,\ B\Big] \quad\Leftrightarrow\quad \frac{dL}{dt}=L\cdot B-B\cdot L.$$

As in [15], the following properties hold:

$$H = Trace(L) \quad \text{and} \quad C = \frac{1}{2}Trace(L^2),$$

where H- Hamiltonian function and C- Casimir function.

Table 2: The comparison between the approximate solutions $\bar{x}_1'$ from Eq. (7) and the corresponding numerical solutions for $\beta = 0$ (relative errors: $\epsilon_{x_1'} = |x'_{1_{numerical}} - \bar{x}_1'|$)

t	$x'_{1_{numerical}}$	$\bar{x}_1'$ from Eq. (7)	$\epsilon_{x_1'}$
0	-3.8645 $\cdot 10^{-21}$	8.8817 $\cdot 10^{-16}$	8.8818 $\cdot 10^{-16}$
4/5	0.371963259413	0.372477797312	5.1453 $\cdot 10^{-4}$
8/5	0.696699514599	0.696023892471	6.7562 $\cdot 10^{-4}$
12/5	0.901065461379	0.901471382767	4.0592 $\cdot 10^{-4}$
16/5	0.980364982283	0.980092859963	2.7212 $\cdot 10^{-4}$
4	0.997770087958	0.997861518336	9.1430 $\cdot 10^{-5}$
24/5	0.999859396033	0.999974757114	1.15361 $\cdot 10^{-4}$
28/5	0.999995149208	0.999946428194	4.8721 $\cdot 10^{-5}$
32/5	0.999999902864	0.999935247532	6.4655 $\cdot 10^{-5}$
36/5	0.999999992429	0.999977995910	2.1996 $\cdot 10^{-5}$
8	0.999999993273	1.000005054164	5.0608 $\cdot 10^{-6}$

6 Numerical simulation

In this section, the accuracy and validity of the OHAM technique is proved using a comparison of our approximate solutions with numerical results obtained via the fourth-order Runge-Kutta method for $\beta = 0$.

The convergence-control parameters K, C_i, $i = \overline{1,\ 8}$ are optimally determined by means of the least-square method using the Mathematica 9.0 software.

Observation: If $\bar{x}(t)$ is the approximate analytic solution obtained via Optimal Homotopy Asymptotic Method [13], then for $\beta = 0$ the convergence-control parameters are respectively :

$$C_1 = -5.146692834756\ ,\ C_2 = -3.319352427903\ ,\ C_3 = 1.365481026558\ ,$$

$$C_4 = -0.109053890316\ ,\ C_5 = 63.014570679440\ ,\ C_6 = -183.226725640327\ ,$$

$$C_7 = 47.317886321776\ ,\ C_8 = 53.798097627583\ ,\ K = 1.679601787261\ .$$

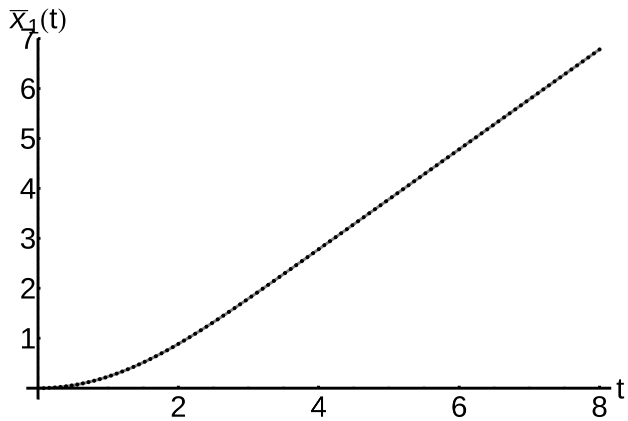

Figure 1: Comparison between the approximate solutions $\bar{x}_1$ given by Eq. (7) and the corresponding numerical solutions:
———— numerical solution, ··········· OHAM solution.

Table 3: The comparison between the approximate solutions $\bar{x}_1''$ from Eq. (7) and the corresponding numerical solutions for $\beta = 0$ (relative errors: $\epsilon_{x_1''} = |x_{1_{numerical}}'' - \bar{x}_1''|$)

t	$x_{1_{numerical}}''$	$\bar{x}_1''$ from Eq. (7)	$\epsilon_{x_1''}$
0	0.469599995897	0.469599895897	$1.0000 \cdot 10^{-7}$
4/5	0.451190185801	0.460093505720	$8.9033 \cdot 10^{-3}$
8/5	0.342486827279	0.342756074406	$2.6924 \cdot 10^{-4}$
12/5	0.167560529122	0.167075663254	$4.8486 \cdot 10^{-4}$
16/5	0.046370185755	0.046300824641	$6.9361 \cdot 10^{-5}$
4	0.006874039262	0.007295190897	$4.2115 \cdot 10^{-4}$
24/5	0.000538393988	0.000306981990	$2.3141 \cdot 10^{-4}$
28/5	0.000022211398	$-9.0012 \cdot 10^{-5}$	$1.1222 \cdot 10^{-4}$
32/5	$4.7872 \cdot 10^{-7}$	$4.2932 \cdot 10^{-5}$	$4.2454 \cdot 10^{-5}$
36/5	$4.0075 \cdot 10^{-9}$	$4.9247 \cdot 10^{-5}$	$4.9243 \cdot 10^{-5}$
8	$6.8538 \cdot 10^{-10}$	$1.8697 \cdot 10^{-5}$	$1.8696 \cdot 10^{-5}$

The first-order approximate solutions proposed in [13] becomes:

$$\bar{x}_1(t) = -1.216776769791 - 0.236541146259 \cdot e^{-6.7184071149045t} + t +$$
$$+e^{-5.0388053617784t} \cdot (2.067180899886 + 0.318125112302t - 0.554796671887t^2) +$$
$$+e^{-3.3592035745221t} \cdot (-4.570156113663 - 5.624329520194t - 8.333219504464t^2 -$$
$$-4.761785285840t^3) + e^{-1.6796017872611t} \cdot (3.956293129828 + 4.426059140587t -$$
$$-1.432577756121t^2 - 0.407499225829t^3 + 0.162858423313t^4 - 0.012985684004t^5)\ . \tag{7}$$

Finally, Tables 1 - 3 and Figs. 1-2 emphasize the accuracy of the OHAM technique by comparing the approximate analytic solutions $\bar{x}_1$, $\bar{x}_1'$ and $\bar{x}_1''$ respectively presented above with the corresponding numerical integration values.

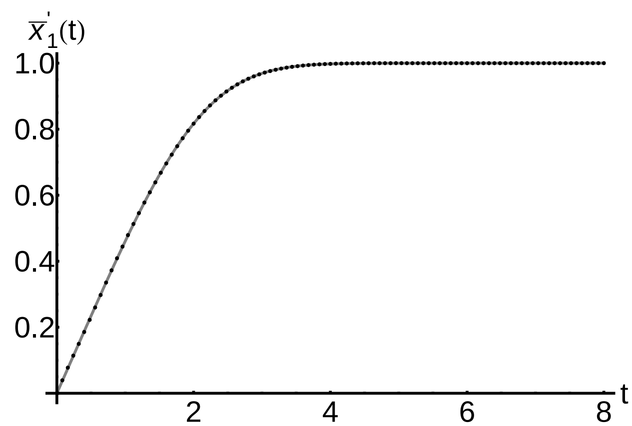

Figure 2: Comparison between the approximate solutions $\bar{x}_1'$ from Eq. (7) and the corresponding numerical solutions: ———— numerical solution, $\cdots\cdots\cdots$ OHAM solution.

7 Conclusion

In this paper we analyze the Falkner-Skan equations from some geometrical point of view. The stability of a nonlinear differential problem governing the Falkner-Skan equation is investigated. Finding a Hamilton-Poisson realization, the results were obtained using specific tools, such as the energy-Casimir method. We give find a Lax formulation for the studied system.

Finally, the analytical integration of the nonlinear system (obtained via the Optimal Homotopy Asymptotic Method and presented in [13]) is compared with the exact solution (obtained as intersections of the surfaces $H(x_1, x_2, x_3) = const.$ and $C(x_1, x_2, x_3) = const$).

Numerical integration of the controlled dynamics is obtained via the Optimal Homotopy Asymptotic Method. Numerical simulations and a comparison with Runge-Kutta 4 steps integrator are presented, too.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. de Vicente (editor), *Viscoelsticity - From theory to biological applications*, http://dx.doi.org/10.5772/3188 (online www.interchopen.com).

[2] V. M. Falkner, S. W. Skan, *Some approximate solutions of the boundary layer equations*, Phil. Magazine, 12, 1931, 865–816.

[3] D. R. Hartree, *On the equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer*, Proc. Cambr. Phil. Soc. 33, 1937, 223–239.

[4] K. Stewartson, *Further solutions of the Falkner-Skan equation*, Proc. Cambr. Phil. Soc., 50, 1954, 454–465.

[5] S. P. Hastings, *Reversed flow solutions of the Falkner-Skan equation*, SIAM, Journal of Applied Mathematics, 22 (2), 1972, 329–334.

[6] E. F. F. Botta, F. J. Hut, A. E. P. Veldman, *The role of periodic solution in the Falknar-Skan problem for* $\lambda \geq 0$, Journal of Engineering Mathematics, 20(1), 1986, 81–93.

[7] M. B. Zaturska, W. N. Banks, *A new method of the Falkner-Skan equation*, Acta Mechanicca , 152, 2001, 197–201.

[8] N. S. Elgazery, *Numerical solution for the Falkner-Skan equation*, Chaos, Solitons and Fractals, 35, 2008, 738–746.

[9] E. Alizadeh, M. Farhadi, K. Sedighi, N. R. E. Kebria, A. Ghafourian, *Solution of the Falkner-Skan equation for wedge by Adomian decomposition method*, Communications in Nonlinear Science and Numerical Simulation, 14, 2009, 724–733.

[10] S. Abbasbandy, T. Hayat, *Solution of the MHD Falkner-Skan flow by Hankel-Padé method*, Physics Letters A, 373 (3), 2009, 731–734.

[11] V. Marinca, N. Herişanu, *Nonlinear Dynamical Systems in Engineering - Some Approximate Approaches*, Springer Verlag, Heidelberg, 2011.

[12] V. Marinca, N. Herişanu, C. Bota, B. Marinca, *An optimal homotopy asymptotic method applied to the steady flow of a fourth grade fluid past a porous plate*, Applied Mathematics Letters, 22, 2009, 245–251.

[13] V. Marinca, R.-D. Ene, B. Marinca, *Analytic Approximate Solution for Falkner-Skan Equation*, Sci. World J., 2014, Volume 2014, Article ID 617453, 22 pages.

[14] M. Puta, *Hamiltonian Systems and Geometric Quantization, Mathematics and Its Applications*, Springer-Verlag, Berlin, vol. 260, 1993.

[15] M. Puta, *Integrability and geometric prequantization of the Maxwell-Bloch equations*, Bull. Sci. Math., 122, 1998, 243–250.

Camelia Petrişor, Remus-Daniel Ene
Department of Mathematics,
Politehnica University Timisoara,
P-ta Victoriei 2, 300006, Timisoara, Romania
E-mail: camelia.petrisor@upt.ro, remus.ene@upt.ro

SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

CONNECTIONS BETWEEN SOME CONCEPTS OF POLYNOMIAL TRICHOTOMY FOR DISCRETE SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES

Diana BORLEA

Abstract

The present paper studies the property of trichotomy described by a polynomial behaviour according to decay, expansion and growth of the solution on the stable, unstable and central manifold respectively. [1]

Keywords and phrases: *skew-evolution semiflow; polynomial trichotomy, strong polynomial trichotomy and weak polynomial trichotomy.*

1 Introduction

The issue of decomposing the state space into a direct sum of subspaces, where the trajectories of the system define a prescribed behavior is triggered by the asymptotic behavior of first-order differential equations. The term of exponential trichotomy shapes the fact that the state space into three closed subspaces: stable subspace, unstable subspace and the so-called central manifold. While the stable subspace leads the pattern of the solution to converge (in norm) towards zero, and the unstable one to converge (in norm) towards infinity, on the central manifold the solutions need only to have polynomially growth and decay.

The trichotomy property is a natural generalization of the well-known dichotomy property of dynamical systems, refined as several results were published from which we point out the following: [1], [4], [5], [6], [7], [8] and [10]. The trichotomy property was first mentioned by Sacker and Sell in [9] and several results, related to polynomial trichotomy, were published in [1], [4], [5], [8], [10].

The present paper studies the property of trichotomy described by a polynomial behavior according to decay, expansion and growth of the solution on the stable, unstable and central manifold respectively. The links between the

[1]MSC (2010): 34D09, 34D05

concepts presented in this paper (polynomial trichotomy, strong polynomial trichotomy and weak polynomial trichotomy) are indicated mostly with the aid of examples and counterexamples, which provide a set of systems having such properties on one hand, and clearly delimiting the concepts on the other hand.

2 Discrete evolution semiflows

Let (X, d) be a metric space, V a Banach space, and $\mathcal{B}(V)$ the Banach space of all bounded linear operators acting on V. We denote by $D = \left\{(m, n) \in \mathbb{N}^2 : m \geq n\right\}$.

Definition 2.1. A mapping $\varphi : D \times X \to X$ is called a *discrete evolution semiflow* on X if the following conditions hold:

(es1) $\varphi(m, m, x) = x$, for all $(m, x) \in \mathbb{N} \times X$;

(es2) $\varphi(m, n, \varphi(n, p, x)) = \varphi(m, p, x)$, for all $(m, n), (n, p) \in D, x \in X$.

Definition 2.2. A mapping $\Phi : D \times X \to \mathcal{B}(V)$ is called a *discrete evolution cocycle* over the evolution semiflow φ if:

(ec1) $\Phi(m, m, x) = I$, for all $m \geq 0, x \in X$.

(ec2) $\Phi(m, n, \varphi(n, p, x))\Phi(n, p, x) = \Phi(m, p, x)$, for all $(m, n), (n, p) \in D$ and for all $x \in X$.

If Φ is a discrete evolution cocycle over the discrete evolution semiflow φ, then the pair $C = (\varphi, \Phi)$, defined by $C : D \times X \times V \to X \times V$, $C(m, n, x, v) = (\varphi(m, n, x), \Phi(m, n, x)v)$ is called a discrete *skew-evolution semiflow* on $X \times V$.

Definition 2.3. An operator valued sequence $P : \mathbb{N} \to \mathcal{B}(V)$ is called a **sequence of projections** if $P_n P_n = P_n$ for all $n \in \mathbb{N}$, where $P_n = P(n)$.

3 Trichotomy cvadruples

We will denote by $V = l^2(\mathbb{N}, \mathbb{R})$ the Banach space containing all the real-valued sequences $v = (v_k)_{k \geq 0}$ having the property

$$\sum_{n=0}^{\infty} |v_n|^2 < \infty,$$

endowed with the norm $\|v\|_2 = \left(\sum\limits_{n=0}^{\infty} |v_n|^2\right)^{1/2}$.

Definition 3.1. A sequence of projections $P : \mathbb{N} \to \mathcal{B}(V)$ is called **polynomially bounded** if there exist $M \geq 1$ and $\gamma \geq 0$ such that $\|P_n\| \leq M(n+1)^\gamma$ for all $n \in \mathbb{N}X$. If $\gamma = 0$, we say that it is **bounded**.

Definition 3.2. Three sequences of projections $P, Q, R : \mathbb{N} \to \mathcal{B}(V)$ are called **supplementary** if for all $n \in \mathbb{N}$ we have $P_n + Q_n + R_n = I$.

In what follows, we will present two examples which will serve our main aim.

Example 3.1. Consider $V = l^2(\mathbb{N}, \mathbb{R})$ and $p : \mathbb{N} \to \mathbb{R}$ a non-decreasing sequence. For each $n \in \mathbb{N}$ we define $P_{1,n} : l^2(\mathbb{N}, \mathbb{R}) \to l^2(\mathbb{N}, \mathbb{R})$ by $P_{1,n}v = (y_k(n))_{k\geq 0}$, where $y_{3k}(n) = v_{3k} + p(n) \cdot v_{3k+1}$, $y_{3k+1}(n) = y_{3k+2}(n) = 0$, $k \in \mathbb{N}$.

We have that

$$P_{1,n} \in \mathcal{B}(l^2(\mathbb{N}, \mathbb{R}))$$

and for all $n \in \mathbb{N}$ we have that

$$\max\{1, p(n)\} \leq \|P_{1,n}\| \leq 1 + p(n).$$

Furthermore, we define the sequence of projections

$$Q_1 : \mathbb{N} \to \mathcal{B}(V)$$

by

$$Q_{1,n}v = (z_k(n))_{k\geq 0},$$

where

$$z_{3k}(n) = -p(n)v_{3k+1},\ \ z_{3k+1}(n) = v_{3k+1},\ \ z_{3k+2}(n) = 0,\ \ k \in \mathbb{N}.$$

The following hold:

$$|Q_{1,n}v\|_2 \leq \|Q_{1,m}v\|_2$$

$$\|Q_{1,n}v\|_2 = \sqrt{(1 + p(n)^2) \cdot \sum_{k=0}^{\infty} |v_{3k+1}|^2}.$$

Finally, we define

$$R_1 : \mathbb{N} \to \mathcal{B}(V)$$

by

$$R_{1,n}v = (w_k(n))_{k\geq 0},$$

where

$$w_{3k}(n) = w_{3k+1}(n) = 0, \quad w_{3k+2}(n) = v_{3k+2},\ \ k \in \mathbb{N}.$$

We have that R_1 is bounded, with $\|R_{1,n}\| = 1$, for all $n \in \mathbb{N}$, $x \in X$ and in addition, the sequences P_1, Q_1 and R_1 are supplementary.

Example 3.2. Let $V = l^2(\mathbb{N},\mathbb{R})$ and consider

$$P_2, Q_2, R_2 : \mathbb{N} \to \mathcal{B}(l^2(\mathbb{N},\mathbb{R}))$$

defined by $P_{2,n}v = (y_k(n))_{k\geq 0}$, $Q_{2,n}v = (z_k(n))_{k\geq 0}$ and $R_{2,n}v = (w_k(n))_{k\geq 0}$, where, for $k \in \mathbb{N}$, $y_{4n}(n) = v_{4k}$, $y_{4k+1}(n) = y_{4k+2}(n) = y_{4k+3}(n) = 0$, $z_{4k}(n) = z_{4k+3}(n) = 0$, $z_{4k+1}(n) = v_{4k+1}$, $z_{4k+2}(n) = v_{4k+2}$, $w_{4k}(n) = w_{4k+1}(n) = w_{4k+2}(n) = 0$, $w_{4k+3}(n) = v_{4k+3}$. We have that P_2, Q_2 and R_2 are three supplementary sequences of projections, with $\|P_{2,n}\| = \|Q_{2,n}\| = \|R_{2,n}\| = 1$ for all $n \in \mathbb{N}$.

Given three supplementary sequences of projections P, Q, R and $C = (\Phi, \varphi)$ a discrete skew-evolution semiflow, we will say that (C, P, Q, R) is a **trichotomic cvadruple**.

Two examples of trichotomic cvadruples are given below.

Example 3.3. On $V = l^2(\mathbb{N},\mathbb{R})$ consider the sequences of projections P_1, Q_1 and R_1 from Example 3.1. Let

$$\lambda : \mathbb{N} \to (0, \infty)$$

and

$$\Phi_1 : D \to \mathcal{B}(l^2(\mathbb{N},\mathbb{R}))$$

given by

$$\Phi_1(m,n,x) = \frac{\lambda(n)}{\lambda(m)} \cdot P_{1,n} + \frac{\lambda(m)}{\lambda(n)} \cdot Q_{1,m} + R_{1,n}$$

for all $(m,n,x) \in D \times X$. Taking into account that, for all $m, n \in \mathbb{N}$ the following hold:

$$P_{1,m}P_{1,n} = P_{1,n} \quad \text{and} \quad Q_{1,m}Q_{1,n} = Q_{1,m}$$

it is easy to check that Φ_1 is a discrete skew-evolution co-cycle. Moreover we have that for all $(m,n,x) \in D \times X$,

$$\Phi_1(m,n,x)P_{1,n} = \frac{\lambda(n)}{\lambda(m)} P_{1,n},$$

$$\Phi_1(m,n,x)Q_{1,n} = \frac{\lambda(m)}{\lambda(n)} Q_{1,m},$$

$$\Phi_1(m,n,x)R_{1,n} = R_{1,n}.$$

Example 3.4. On $V = l^2(\mathbb{N}, \mathbb{R})$ let P_2, Q_2 and R_2 be the sequences of projections defined in Example 3.2. For $\psi : \mathbb{N} \to (0, \infty)$ we define

$$\Phi_2 : D \to \mathcal{B}(l^2(\mathbb{N}, \mathbb{R}))$$

by

$$\Phi_2(m, n, x)v = \begin{cases} (y_k(m, n))_{k \geq 0} & \text{if } m > n \\ v, & \text{if } m = n \end{cases}$$

where for all $k \in \mathbb{N}$ and $(m, n, x, v) \in D \times X \times l^2(\mathbb{N}, \mathbb{R})$,

$$y_{4k}(m, n) = \frac{\psi(n)}{\psi(m)} v_{4k},$$

$$y_{4k+1}(m, n) = \frac{\psi(m)}{\psi(n)} v_{4k+1},$$

$$y_{4k+2}(m, n) = 0$$

and

$$y_{4k+3}(m, n) = v_{4k+3}.$$

One can easily observe that (Φ_2, P_2, Q_2, R_2) a trichotomic cvadruple and for

$$(m, n, x) \in D \times l^2(\mathbb{N}, \mathbb{R})$$

we have that

$$\Phi_2(m, n, x)P_{2,n}v = (p_k(m, n))_{k \geq 0},$$

where

$$p_{4k}(m, n) = \frac{\psi(n)}{\psi(m)} v_{4k},$$

$$p_{4k+1}(m, n) = p_{4k+2}(m, n) = p_{4k+3}(m, n) = 0$$

and

$$\Phi_2(m, n, x)Q_{2,n}v = \begin{cases} (q_k(m, n))_{k \geq 0}, & m > n \\ (\rho_k(m, n))_{k \geq 0}, & m = n \end{cases}$$

is given by

$$q_{4k}(m,n) = q_{4k+2}(m,n) = q_{4k+3}(m,n) = 0,$$

$$q_{4k+1}(m,n) = \frac{\psi(m)}{\psi(n)} v_{4k+1}$$

and

$$\rho_{4k}(m,n) = \rho_{4k+3}(m,n) = 0,$$

$$\rho_{4k+1}(m,n) = v_{4k+1},$$

$$\rho_{4k+2}(m,n) = v_{4k+2},$$

for all $n \in \mathbb{N}$, and

$$\Phi_2(m,n,x)R_2(n)v = (r_k(m,n))_{k\geq 0},$$

where

$$r_{4k}(m,n) = r_{4k+1}(m,n) = r_{4k+2}(m,n) = 0,$$

$$r_{4k+3}(m,n) = v_{4k+3}.$$

In what follows, we will present the main concepts of trichotomy, which will be studied and delimited in the remaining sections.

4 Concepts of discrete polynomial trichotomy

Definition 4.1. A trichotomic cvadruple (C,P,Q,R) is called **polynomiallty trichotomic** (p.t) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that for all $(m,n,x) \in D \times X$,

(pt$_1$) $(m+1)^\alpha \|\Phi(m,n,x)P_n\| \leq N(n+1)^{\alpha+\beta}$;

(pt$_2$) $(m+1)^\alpha \leq N(m+1)^\beta (n+1)^\alpha \|\Phi(m,n,x)Q_n\|$;

(pt$_3$) $(n+1)^\alpha \|\Phi(m,n,x)R_n\| \leq N(m+1)^\alpha (n+1)^\beta$;

(pt$_4$) $(n+1)^{\alpha} \leq N(m+1)^{\alpha+\beta}\|\Phi(m,n,x)R_n\|$.

If $\beta = 0$, then we say that (C, P, Q, R) is **uniformly polynomially trichotomic** (u.p.t).

Remark 4.1. If (C, P, Q, R) is (p.t) with constants N, α, β then

$$\max\{\|P_n\|, \|Q_n\|, \|R_n\|\} \leq 3N(n+1)^{\beta}, \quad \forall n \in \mathbb{N}.$$

Remark 4.2. If (C, P, Q, R) is (u.p.t) then it is also (p.t). The converse is not generally true. Consider, for example, the trichotomic cvadruple (Φ_1, P_1, Q_1, R_1) from Example 3.3 with $p(n) = \lambda(n) = n+1$. It is easy to check that (Φ_1, P_1, Q_1, R_1) is (p.t), but it cannot be (u.p.t), because P is not bounded.

Definition 4.2. A trichotomic cvadruple (C, P, Q, R) is said to be **strongly polynomially trichotomic** (s.p.t) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

(spt$_1$) $(m+1)^{\alpha}\|\Phi(m,n,x)P_n v\| \leq N(n+1)^{\alpha+\beta}\|P_n v\|$;

(spt$_2$) $(m+1)^{\alpha}\|Q_n v\| \leq N(m+1)^{\beta}(n+1)^{\alpha}\|\Phi(m,n,x)Q_n v\|$;

(spt$_3$) $(n+1)^{\alpha}\|\Phi(m,n,x)R_n v\| \leq N(m+1)^{\alpha}(n+1)^{\beta}\|R_n v\|$;

(spt$_4$) $(n+1)^{\alpha}\|R_n v\| \leq N(m+1)^{\alpha+\beta}\|\Phi(m,n,x)R_n v\|$

for all $(m, n, x, v) \in D \times X \times V$.

If $\beta = 0$, then we say that (C, P, Q, R) is **uniformly strongly polynomially trichotomic** (u.s.p.t).

Remark 4.3. If (C, P, Q, R) is (u.s.p.t) then it is also (s.p.t). The converse is not generally true, fact shown by Example 5.1.

Remark 4.4. If (C, P, Q, R) is (s.p.t) then for all $(m, n, x) \in D \times X$ one has that $Range\ Q_n \cap Ker\ \Phi(m,n,x) = Range\ R_n \cap Ker\ \Phi(m,n,x) = \{0\}$.

Definition 4.3. A trichotomic cvadruple (C, P, Q, R) is said to be **weakly polynomially trichotomic** (w.p.t) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

(wpt$_1$) $(m+1)^{\alpha}\|\Phi(m,n,x)P_n\| \leq N(n+1)^{\alpha+\beta}\|P_n\|$;

(wpt$_2$) $(m+1)^{\alpha}\|Q_n\| \leq N(m+1)^{\beta}(n+1)^{\alpha}\|\Phi(m,n,x)Q_n\|$;

(wpt$_3$) $(n+1)^{\alpha}\|\Phi(m,n,x)R_n\| \leq N(m+1)^{\alpha}(n+1)^{\beta}\|R_n\|$;

(wpt$_4$) $(n+1)^{\alpha}\|R_n\| \leq N(m+1)^{\alpha+\beta}\|\Phi(m,n,x)R_n\|$

for all $(m,n,x) \in D \times X$.

If $\beta = 0$ then we say that (C,P,Q,R) is **uniformly weakly polynomially trichotomic** (u.w.p.t).

Remark 4.5. If (C,P,Q,R) is (u.w.p.t) then it is also (w.p.t). The converse is not generally true, fact illustrated by Example 5.2.

In what follows, the connections between the above defined concepts are presented.

Remark 4.6. If a trichotomic cvadruple (C,P,Q,R) is (s.p.t) then it is also (w.p.t). Moreover, if (C,P,Q,R) is (u.s.p.t), then it is also (u.w.p.t).

Proposition 4.1. *Let (C,P,Q,R) be a trichotomic cvadruple. If (C,P,Q,R) is (p.t) then it is also (w.p.t). Moreover, (u.p.t) $\Rightarrow$ (u.w.p.t).*

Proof. It follows the reasoning from Proposition 3.11 from [2]. $\square$

Remark 4.7. Example 5.3 shows that (s.p.t) does not imply (p.t) and (u.s.p.t) does not imply (u.p.t). Example 5.4 shows that the concepts of (p.t) and (w.p.t) do not coincide. Example 5.5 shows that (p.t) doesn't imply (s.p.t) and (u.p.t) doesn't imply (u.s.p.t). Finally, Example 5.6 shows that (w.p.t) doesn't imply (s.p.t) and (u.w.p.t) doesn't imply (u.s.p.t).

Remark 4.8. The connections between the above enumerated concepts, taking into account the presented results, and the examples from the next section, are illustrated by the following diagram:

$$\begin{array}{ccccccc}
\boxed{\text{u.p.t}} & \overset{\nLeftarrow}{\nRightarrow} & \boxed{\text{u.s.p.t}} & \overset{\nLeftarrow}{\Rightarrow} & \boxed{\text{u.w.p.t}} & \overset{\Leftarrow}{\nRightarrow} & \boxed{\text{u.p.t}} \\
\not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow \\
\boxed{\text{p.t}} & \overset{\nLeftarrow}{\nRightarrow} & \boxed{\text{s.p.t}} & \overset{\nLeftarrow}{\Rightarrow} & \boxed{\text{w.p.t}} & \overset{\Leftarrow}{\nRightarrow} & \boxed{\text{p.t}}
\end{array}$$

5 Examples and counterexamples

Example 5.1. We will consider a simplified example. On $V = \mathbb{R}^3$, endowed with the canonical norm, consider $P,Q,R : \mathbb{N} \to \mathcal{B}(V)$ the sequences of constant canonical projections on $\mathbb{R}^3$, on the first, second and third coordinate respectively. We define, for all $(m,n,x) \in D \times X$:

$$\Phi(m,n,x) = \frac{(n+1)^{1+a_n}}{(m+1)^{1+a_m}}P_n + \frac{m+1}{n+1}Q_n + R_n,$$

where $a_n = \chi_{2\mathbb{N}+1}(n)$, $n \in \mathbb{N}$ (χ_A denotes the characteristic function of the set A). It is easy to see that (Φ, P, Q, R) is a trichotomic cvadruple which is (s.p.t) with $N = \alpha = \beta = 1$. But, if we would assume that (Φ, P, Q, R) is (u.s.p.t), then, in particular, there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m, n, x, v) \in D \times X \times V$, we have that

$$(m+1)^{\alpha} \|\Phi(m, n, x) P_n v\| \leq N(n+1)^{\alpha} \|P_n v\|.$$

Let

$$v = (1, 0, 0) \in Range P_n$$

and $k \in \mathbb{N}$. Fix $x \in X$ and choose $m = 2k+2$ and $n = 2k+1$. The above inequality yields the following contradiction:

$$2k + 2 \leq N \left(\frac{2k+2}{2k+3} \right)^{\alpha - 1}$$

for all $k \in \mathbb{N}$.

Example 5.2. Let (Φ_1, P_1, Q_1, R_1) be as in Example 5.1. According to Remark 4.6 we have that (Φ_1, P_1, Q_1, R_1) is (w.p.t). The same contradiction is obtained, as in Example 5.1, by assuming that (Φ_1, P_1, Q_1, R_1) is (u.w.p.t).

Example 5.3. Let (Φ_1, P_1, Q_1, R_1) the trichotomic cvadruple from Example 3.3 cu $p(n) = (n+1)^{n+1}$ and $\lambda(n) = n+1$. From the following estimations

$$(m+1)\|\Phi_1(m, n, x) P_{1,n} v\|_2 = (n+1)\|P_{1,n} v\|_2$$

$$(m+1)\|Q_{1,n} v\|_2 \leq \lambda(m)\|Q_{1,m} v\|_2 = (n+1)\|\Phi_1(m, n, x) Q_{1,n} v\|_2$$

$$(n+1)\|\Phi(m, n, x) R_n v\|_2 \leq (n+1)(m+1)\|R_n v\|_2$$

$$(n+1)\|R_n v\|_2 \leq N(m+1)^2 \|\Phi(m, n, x) R_n v\|_2$$

valid for all $(m, n, x, v) \in D \times X \times V$, we can see that (Φ_1, P_1, Q_1, R_1) is (u.s.p.t), hence it is also (s.p.t).
Assume by a contradiction that (Φ_1, P_1, Q_1, R_1) is (p.t). Then, according to Remark 4.1, we have that there exist $M \geq 1$, $\gamma \geq 0$ such that $\|P_{1,n}\| \leq M(n+1)^{\gamma}$, or all $n \in \mathbb{N}$. This leads us to $(n+1)^{n+1} = p(n) \leq \|P_{1,n}\| \leq M(n+1)^{\gamma}$. We conclude that (Φ_1, P_1, Q_1, R_1) is not (p.t) hence not (u.p.t) as well.

Example 5.4. Let (Φ_1, P_1, Q_1, R_1) the trichotomic cvadruple from Example 5.3. According to Remark 4.6, we have that (Φ_1, P_1, Q_1, R_1) is (u.w.p.t), hence it is also (w.p.t). But, by Example 5.3, it is not (p.t), nor (u.p.t).

Example 5.5. Let (C, P_2, Q_2, R_2) the trichotomic cvadruple from Example 3.4 with $\psi(n) = n + 1$. This leads us easily to the fact that that (C, P_2, Q_2, R_2) is (u.p.t).
In what follows, we will show that (C, P_2, Q_2, R_2) is not (s.p.t), and from here, it cannot be neither (u.s.p.t). Assume, by a contradiction, that (C, P_2, Q_2, R_2) is (s.p.t). Let $v = (v_k)_{k\geq 0}$ given by $v_{4k+2} = \frac{1}{4k+2}$, $v_{4k+3} = v_{4k+1} = v_{4k} = 0$, $k \in \mathbb{N}$. Obviously $v \in l^2(\mathbb{N}, \mathbb{R})$ and by denoting, for every $n \in \mathbb{N}$, $Q_{2,n}v = (z_k(n))_{k\geq 0}$, where $z_{4k}(n) = z_{4k+1}(n) = x_{4k+1} = z_{4k+3} = 0,\ z_{4k+2}(n) = v_{4k+2} = \frac{1}{4k+2}$, we can easily see that $(z_k(n))_{k\geq 0}$ is a nonzero sequence. Let now $(m, n, x) \in D \times X$ be with $m > n$. By denoting

$$\Phi_2(m, n, x)Q_{2,n}v = (q_k(m, n))_{k\geq 0},$$

with

$$q_{4k}(m, n) = q_{4k+1}(m, n) = \frac{m + 1}{n + 1} v_{4k+1} = q_{4k+2}(m, n) = q_{4k+3}(m, n) = 0,$$

it follows that $\Phi_2(m, n, x)Q_{2,n}v = 0$, which contradicts the facts proven in Remark 4.4, hence (C, P_2, Q_2, R_2) is not (s.p.t).

Example 5.6. Let (C, P_2, Q_2, R_2) the trichotomic cvadruple from Example 5.5. Taking into account that for all $n \geq 0$,

$$\|P_{2,n}\| = \|Q_{2,n}\| = \|R_{2,n}\| = 1,$$

it follows that (C, P_2, Q_2, R_2) is (u.w.p.t), hence (w.p.t). Again, by Example 5.5, we obtain that (C, P_2, Q_2, R_2) is not (s.p.t), hence it is neither (u.s.p.t).

References

[1] M.-G. Babuţia, T. Ceauşu, N. M. Seimeanu, *On Uniform Exponential Dichotomy of Evolution operators*, An. Univ. Timişoara Ser. Math. Inform. L, 2 (2012), 3-14.

[2] M.-G. Babuţia, N. M. Seimeanu, T. Ceauşu, *Polinomyal dichotomies for evolution operators through examples and counterexamples*, Bull. Instit. Politehn. Iaşi, Tom LX (LXIV), Fasc, 4, 2014, s31-52.

[3] D. Borlea, *Polynomial trichotomy for skew-evolution semiflows on Banach spaces*, Proceedings of the International Symposium - Research and Education in Innova Era, Section Mathematics and Computer Science, 6th Edition, Aurel Vlaicu University of Arad Publishing House (2016), 38-47.

[4] L. Barreira, C. Valls, *Polynomial growth rates*, Nonlinear Anal.-Theor. 71 (2009), 5208-5219.

[5] L. Barreira, M. Fan, C. Valls, J. Zhang, *Robustness of nonuniform polynomial dichotomies for difference equations*, Topol. Methods Nonlinear Anal. 37 (2011), no. 2, 357-376.

[6] M. I. Kovacs, M. G. Babutia, M. Megan, *On (h,k)-dichotomy in Banach spaces* , Bull. Ştiinţ Univ. Politehn. Timiş. Ser. Mat.-Fiz. 59 (73) (2014), Fasc. 1/2014, 19-27.

[7] M. Megan, A. L. Sasu, B. Sasu, *Discrete admissibility and exponential dichotomy for evolution families*, Discrete Contin. Dynam. Systems 9 (2003), 383-397.

[8] M. L. Rămneanţu, T. Ceauşu, M. Megan, *On nonuniform polynomial dichotomy of evolution operators in Banach spaces*, Int. J. Pure Appl. Math. 75 (2012), 305-318.

[9] R. J. Sacker, G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems*, J. Diff. Equat. 15(3) (1974), 429-258.

[10] A. L. Sasu, B. Sasu, *Discrete admissibility and exponential trichotomy of dynamical systems*, Discrete Contin. Dyn. Syst. 34 (7) (2014), 2929-2962.

Diana Borlea
West University of Timişoara,
Department of Mathematics,
V. Pârvan Blv. No. 4, 300223 - Timişoara, Romania,
"Vasile Goldiş" Western University of Arad,
M. Eminescu Str. No. 15, 310086 - Arad, Romania,
dianab268@yahoo.com

SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

ON UNIFORM POLYNOMIAL DICHOTOMY IN BANACH SPACES

Rovana BORUGA, Mihail MEGAN

Abstract

The main objective of the present paper is to describe the polynomial dichotomy behaviour in the uniform case of evolution operators in Banach spaces. In this sense we generalize the uniform polynomial stability notion by giving necessary and sufficient conditions for the dichotomy concept. [1]

Keywords and phrases: *evolution operator, uniform polynomial dichotomy*

1 Introduction

The concept of exponential dichotomy was introduced in 1930 by O. Perron [4] and it has been studied for many years. Even though nowadays it plays an important role in the theory of dynamical systems, there are some situations in which the notion of exponential dichotomy is too restrictive for the dynamics and for this reason it is important to have in mind a more general type of dichotomic behavior. In this sense, we refer to the polynomial dichotomy notion, which was firstly mentioned for the nonuniform case by Barreira and Valls in [1]. Moreover, the are many other works that deal with the polynomial asymptotic behaviors of evolution operators [2], [3], [5].
The aim of this paper is to give characterization theorems for the uniform polynomial dichotomy concept. The obtained results generalizes some well-known theorems given for the stability property.

2 Preliminaries

Let X be a real or complex Banach space and $\mathrm{B}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms on X and on $\mathrm{B}(X)$ will be denoted by $\|.\|$. The identity operator on X is denoted by I. We also denote by

[1]MSC (2010): *34D05, 34D09*

$$\Delta = \{(t,s) \in R_+^2 : t \geq s\}, \qquad \Delta_1 = \{(t,s) \in \Delta : s \geq 1\}$$

and

$$T = \{(t,s,t_0) \in R_+^3 : t \geq s \geq t_0\}, \qquad T_1 = \{(t,s,t_0) \in T : t_0 \geq 1\}.$$

Definition 2.1. *An application $U : \Delta \to B(X)$ is said to be an evolution operator on X if*

(e_1) $U(t,t) = I$ *for every* $t \geq 0$

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ *for all* $(t,s,t_0) \in T$.

Definition 2.2. *An evolution operator $U : \Delta \to B(X)$ is said to be* strongly measurable *if for all $(s,x) \in R_+ \times X$, the mapping $t \mapsto \|U(t,s)x\|$ is measurable on $[s,\infty)$.*

Definition 2.3. *An application $P : R_+ \to B(X)$ is said to be a projection family on X if $P^2(t) = P(t)$, for all $t \geq 0$.*

Remark 2.1. *If $P : R_+ \to B(X)$ is a projection family on X, then the mapping $Q : R_+ \to B(X), Q(t) = I - P(t)$ is also a projection family on X, which is called the complementary projection of P.*

Definition 2.4. *A projection family $P : R_+ \to B(X)$ is said to be invariant to the evolution operator $U : \Delta \to B(X)$ if*

$$U(t,s)P(s) = P(t)U(t,s),$$

for all $(t,s) \in \Delta$.

In what follows, if $P : R_+ \to \mathrm{B}(X)$ is an invariant projection family to the evolution operator $U : \Delta \to \mathrm{B}(X)$, we will say that (U,P) is a dichotomic pair.

Definition 2.5. *The pair (U,P) is* uniformly polynomially dichotomic (u.p.d.) *if there are $N \geq 1$ and $\nu > 0$ such that:*

(upd_1) $(t+1)^\nu \|U(t,s)P(s)x\| \leq N(s+1)^\nu \|P(s)x\|$

(upd_2) $(t+1)^\nu \|Q(s)x\| \leq N(s+1)^\nu \|U(t,s)Q(s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.6. *The pair (U,P) is* uniformly logarithmic dichotomic (u.l.d.) *if there exists $L > 1$ such that:*

(uld_1) $\|U(t,s)P(s)x\| \ln \frac{t+1}{s+1} \leq L\|P(s)x\|$

(uld_2) $\|Q(s)x\| \ln \frac{t+1}{s+1} \leq L\|U(t,s)Q(s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.7. *The pair* (U,P) *is* uniformly dichotomic (u.d.) *if there exists* $N \geq 1$ *such that*

(ud_1) $\|U(t,s)P(s)x\| \leq N\|P(s)x\|$

(ud_2) $\|Q(s)x\| \leq N\|U(t,s)Q(s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.8. *The pair* (U,P) has uniform polynomial growth (u.p.g.) *if there are* $M \geq 1$ *and* $\omega > 0$ *such that*

(upg_1) $(s+1)^\omega\|U(t,s)P(s)x\| \leq M(t+1)^\omega\|P(s)x\|$

(upg_2) $(s+1)^\omega\|Q(s)x\| \leq M(t+1)^\omega\|U(t,s)Q(s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Remark 2.2. *It is obvious that*

$$u.p.d. \Rightarrow u.d. \Rightarrow u.p.g.$$

3 Uniform polynomial dichotomy

Lemma 3.1. *Let* $U : \Delta \to B(X)$ *be an evolution operator and* $P : R_+ \to B(X)$ *a projection family invariant to* U. *If* (U,P) *is u.l.d. then there exists* $L > 1$ *such that for all* $(t,s) \in \Delta_1$ *there exists* $n \in \mathbb{N}$ *with the following properties:*

(i) $se^{4nL} \leq t < se^{4(n+1)L}$

(ii) $\|U(se^{4nL},s)P(s)x\| \leq \frac{1}{2^n}\|P(s)x\|$

(iii) $\|U(se^{4nL},s)Q(s)x\| \geq 2^n\|Q(s)x\|, \ \forall x \in X$

Proof. It follows immediately by taking $n = \left[\ln\left(\frac{t}{s}\right)^{\frac{1}{4L}}\right]$.

$\square$

The next theorem is a logarithmic criterion for the uniform polynomial dichotomy concept.

Theorem 3.2. *The pair (U,P) is uniformly polynomially dichotomic if and only if (U,P) has uniform polynomial growth and (U,P) is uniformly logarithmic dichotomic.*

Proof. Necessity. We suppose that (U,P) is u.p.d. Then, from Remark 2.2 we obtain that (U,P) has u.p.g. We prove that (U,P) is u.l.d. We consider the application

$$f:[1,\infty)\to R, f(t)=\frac{\ln t}{t},$$

with $f(t)\le \frac{1}{e}$. Then, for the first condition (uld_1) we have

$$\|U(t,s)P(s)x\|\ln\frac{t+1}{s+1}\le N\left(\frac{s+1}{t+1}\right)^{\nu}\|P(s)x\|\ln\frac{t+1}{s+1}=$$
$$\frac{N}{\nu}\left(\frac{s+1}{t+1}\right)^{\nu}\|P(s)x\|\ln\left(\frac{t+1}{s+1}\right)^{\nu}=$$
$$\frac{N}{\nu}\|P(s)x\|f\left(\left(\frac{t+1}{s+1}\right)^{\nu}\right)\le$$

$$\le\frac{N}{\nu}\cdot f(t)\|P(s)x\|\le\frac{N}{\nu e}\|P(s)x\|.$$

For (uld_2) we do in a similar manner and we obtain

$$\|Q(s)x\|\ln\frac{t+1}{s+1}\le\frac{N}{\nu e}\|U(t,s)Q(s)x\|.$$

So, we have that (U,P) is u.l.d. for $L=\frac{N}{\nu e}+1$.

Sufficiency. Let $N=2Me^{4L\omega}$ and $\nu=\frac{\ln 2}{4L}$.

$$\|U(t,s)P(s)x\|=\|U(t,se^{4nL})U(se^{4nL},s)P(s)x\|\le$$

$$\le M\left(\frac{t+1}{se^{4nL}+1}\right)^{\omega}\|U(se^{4nL}+1,s)P(s)x\|\le$$

$$\le M\cdot e^{4L\omega}\cdot\frac{1}{2^n}\|P(s)x\|=\frac{N}{2^{n+1}}\|P(s)x\|=\frac{N}{e^{(n+1)\ln 2}}\|P(s)x\|\le$$

$$\le\frac{N}{\left(\frac{t+1}{s+1}\right)^{\frac{\ln 2}{4L}}}\|P(s)x\|=N\cdot\left(\frac{s+1}{t+1}\right)^{\nu}\|P(s)x\|$$

It results in the same way as (upd_1).

□

Another characterization of uniform polynomial dichotomy concept is given by

Theorem 3.3. *The pair (U, P) is uniformly polynomially dichotomic if and only if (U, P) has uniform polynomial growth and there exists $r > 1$ such that*

(upH_1) $2\|U(rs, s)P(s)x\| \leq \|P(s)x\|$

(upH_2) $\|U(rs, s)Q(s)x\| \geq 2\|Q(s)x\|$

for all $s \geq 1, x \in X$.

Proof. Necessity We suppose that (U, P) is u.p.d. Then, from Remark 2.2 we obtain that (U, P) has u.p.g. Now, let $r = 2(2N)^{\frac{1}{\nu}}$.

(upH_1)

$$\|U(rs, s)P(s)x\| \leq N\left(\frac{s+1}{rs+1}\right)^{\nu}\|P(s)x\| \leq N \cdot \left(\frac{2}{r}\right)^{\nu}\|P(s)x\|\frac{1}{2}\|P(s)x\|.$$

(upH_2)

$$\|U(rs, s)Q(s)x\| \geq \frac{\|Q(s)x\|}{N}\left(\frac{rs+1}{s+1}\right)^{\nu} \geq \frac{\|Q(s)x\|}{N} \cdot \left(\frac{r}{2}\right)^{\nu} = 2\|Q(s)x\|.$$

Sufficiency Let $(t, s) \in \Delta_1$ and $n = \left[\ln\left(\frac{t}{s}\right)^{\frac{1}{\ln r}}\right]$. Then we obtain the relation $sr^n \leq t < sr^{n+1}$. In order to prove that (U, P) is u.p.d., we show that (U, P) is u.l.d. and then we use Theorem 3.2.

(upL_1)

$$\begin{aligned}
&\|U(t,s)P(s)x\| = \|U(t,sr^n)P(sr^n)U(sr^n,s)P(s)x\| \leq \\
&\leq M\left(\frac{t+1}{sr^n+1}\right)^{\omega}\|P(sr^n)U(sr^n,s)P(s)x\| \leq \\
&\leq M\cdot(r+1)^{\omega}\|U(sr^n,s)P(s)x\| = \\
&= M(r+1)^{\omega}\|U(sr^n,sr^{n-1})P(sr^{n-1})U(sr^{n-1},s)P(s)x\| \leq \\
&\leq \frac{M}{2}(r+1)^{\omega}\|U(sr^{n-1},s)P(s)x\| \leq \cdots \leq \frac{2M(r+1)^{\omega}}{2^{n+1}}\|P(s)x\| \leq \\
&\leq \frac{\ln r}{\ln\frac{t+1}{s+1}}\cdot 2M(r+1)^{\omega}\|P(s)x\|
\end{aligned}$$

(upL_2) We apply the evolution property and we use the same technique as in the previous case. We obtain

$$\|U(t,s)Q(s)x\| \geq \frac{\|Q(s)x\|\cdot\dfrac{1}{\ln r}\cdot\ln\dfrac{t+1}{s+1}}{2M(r+1)^{\omega}}$$

Finally, we have that (U,P) is u.l.d. for $L = 2M(r+1)^{\omega}\ln r + 1$ and from Theorem 3.2, it results that (U,P) is u.p.d. $\square$

Remark 3.4. *The previous theorem is a generalization of some results proved by Hai in [3].*

In what follows, we will present a characterization of Datko type of the uniform polynomial dichotomy concept.

Theorem 3.5. *Let (U,P) be a strongly measurable dichotomic pair with uniform polynomial growth. Then (U,P) is uniformly polynomially dichotomic if and only if there exists $D>1$ with*

(upD_1) $$\int_t^{\infty}\frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1}d\tau \leq D\|U(s,t_0)P(t_0)x_0\|$$

(upD_2) $$\int_{t_0}^{t}\frac{\|U(s,t_0)Q(t_0)x_0\|}{s+1}ds \leq D\|U(t,t_0)Q(t_0)x_0\|$$

for all $(t,t_0,x_0)\in\Delta\times X$.

Proof. Necessity. A simple computation shows that the relations (upD_1) and (upD_2) take place for $D = 1 + \dfrac{N}{\nu}$.

Sufficiency. Step 1. We show that (U, P) is uniformly dichotomic.

(ud_1) If $t \geq 2s + 1$ then

$$\|U(t,t_0)P(t_0)x_0\| = \frac{2}{t+1}\int\limits_{\frac{t-1}{2}}^{t} \|U(t,t_0)P(t_0)x_0\| d\tau \leq$$

$$\leq 2M \int\limits_{\frac{t-1}{2}}^{t} \left(\frac{t+1}{\tau+1}\right)^{\omega} \frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1} d\tau \leq$$

$$\leq DM2^{\omega}\|U(s,t_0)P(t_0)x_0\| = M_1\|U(s,t_0)P(t_0)x_0\|$$

where $M_1 = MD2^{\omega} + 1$.

If $t \in [s, 2s+1)$ then $\dfrac{t+1}{s+1} \leq 2$. We obtain

$$\|U(t,t_0)P(t_0)x_0\| \leq M \left(\frac{t+1}{s+1}\right)^{\omega} \|U(s,t_0)P(t_0)x_0\| \leq$$

$$\leq 2^{\omega}M\|U(s,t_0)P(t_0)x_0\| \leq M_1\|U(s,t_0)P(t_0)x_0\|.$$

(ud_2) Analogous with (ud_1).

Step 2. We prove that U is u.p.d.

(upl_1)

$$\|U(t,t_0)P(t_0)x_0\| \ln \frac{t+1}{s+1} = \int\limits_{s}^{t} \frac{\|U(t,t_0)P(t_0)x_0\|}{\tau+1} d\tau \leq$$

$$\leq M_1 \int\limits_{s}^{t} \frac{\|P(\tau)U(\tau,t_0)x_0\|}{\tau+1} d\tau \leq \; DM_1\|U(s,t_0)P(t_0)x_0\|.$$

(upl_2)

$$\|Q(t_0)x_0\| \ln \frac{t+1}{s+1} = \int\limits_{s}^{t} \frac{\|Q(t_0)x_0\|}{\tau+1} d\tau \leq M_1 \int\limits_{s}^{t} \frac{\|U(\tau,t_0)Q(t_0)x_0\|}{\tau+1} d\tau \leq$$

$$\leq DM_1\|U(t,t_0)Q(t_0)x_0\|.$$

For $t_0 = s,\ x_0 = x$ and from Theorem (3.2) we obtain the conclusion.

□

The next theorem is a characterization which uses Lyapunov functions for the uniform polynomial dichotomy of an evolution operator.

Theorem 3.6. *Let (U,P) be a strongly measurable dichotomic pair with uniform polynomial growth. Then (U,P) is uniformly polynomially dichotomic if and only if there are $D > 1$ and $L : \Delta \times X \to R_+$ with the properties:*

(i) $L(t,t_0,x_0) \le D\left(\|U(t,t_0)P(t_0)x_0\| + \|U(t,t_0)Q(t_0)x_0\|\right)$
for all $(t,t_0,x_0) \in \Delta \times X$

(ii) $$L(t,t_0,P(t_0)x_0) + \int_s^t \frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1}d\tau = L(s,t_0,P(t_0)x_0)$$
for all $(t,s,t_0,x_0) \in T \times X$

(iii) $$L(s,t_0,Q(t_0)x_0) + \int_s^t \frac{\|U(\tau,t_0)Q(t_0)x_0\|}{\tau+1}d\tau = L(t,t_0,Q(t_0)x_0),$$
for all $(t,s,t_0,x_0) \in T \times X$.

Proof. Necessity. If U is u.p.d. then by Theorem (3.5) the function

$$L : \Delta \times X \to R_+$$

defined by

$$L(t,t_0,x_0) = \int_s^\infty \frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1}d\tau + \int_{t_0}^t \frac{\|U(\tau,t_0)Q(t_0)x_0\|}{\tau+1}d\tau$$

satisfies the conditions $(i) - (iii)$.

Sufficiency.
It follows from Theorem (3.5). □

References

[1] L. Barreira, C. Valls, Polynomial growth rates, *Nonlinear Anal.*, **71**, (2009), 5208-5219.

[2] R. Boruga, M. Megan, On some concepts regarding the polynomial behaviors for evolution operators in Banach spaces, *Proceedings of International Symposium "Research and Education in Innovation Era", 7th Edition, Mathematics and Computer Science "Aurel Vlaicu" University of Arad Publishing House* (2018) ISSN 2065 2569, 18-24.

[3] P. V. Hai, On the polynomial stability of evolution families, *Applicable Analysis*, **95(6)** , (2015), 1239-1255.

[4] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, *Math. Z.*, **32**, (1930), 703-728.

[5] M. L. Rămneanţu, Uniform polynomial dichotomy of evolution operators in Banach spaces, *Analele Universităţii din Timişoara Ser. Mat.-Inform.* , **49(1)**, (2011), 107-116.

Rovana Boruga – Department of Mathematics,
West University of Timişoara,
V. Parvan Blvd., No. 4, 300223, Timişoara, ROMANIA
E-mail: rovanaboruga@gmail.com

Mihail Megan – Department of Mathematics,
Academy of Romanian Scientists
Independenţei 54, 050094 Bucharest, Romania
West University of Timişoara
V. Parvan Blvd., No. 4, 300223, Timişoara, ROMANIA
E-mail: mihail.megan@gmail.com

SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

SOLVING FRACTIONAL ORDINARY DIFFERENTIAL EQUATION USING PLSM

Mădălina - Sofia PAŞCA, Marioara LĂPĂDAT

Abstract

In this paper, we obtaining analytical approximate solutions for fractional ordinary differential equations using *Polynomial Least Square Method* (*PLSM*). An example is illustrated to show the presented methods efficiency and convenience. [1]

Keywords and phrases: *Fractional ordinary differential equations, Polynomial Least Square Method(PLSM), Caputos fractional derivative*

1 Introduction

In recent years, fractional ordinary differential equations have been investigated by many authors. Fractional ordinary differential equations are generally used in many branches of science such as: mathematics, physics, chemistry and engineering.

Since most of these equations have no exact solutions, it has been necessary to develop numerical methods or analytical methods to find the approximate solutions of these equations.

In order to find approximate solutions of these equations, many methods were proposed, such as:

- Fractional Adams-Bashforth-Moulton method [2];
- Adomian decomposition method [4];
- Homotopy analysis method [3], [8];
- Variational iteration method [9], [10].

We consider the following fractional ordinary differential equation:

$$D^{\alpha}y(x) = f(x, y(x)) \tag{1}$$

[1]MSC (2010): 60H20, 34F15

$\alpha > 0$, with the initial condition:

$$y(0) = \nu_0 \tag{2}$$

where ν_0 are real constant and D^α denote the Caputo's fractional derivative:

$$D^\alpha \tilde{y}(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_0^x (x-\zeta)^{n-\alpha-1} \cdot \tilde{y}^{(n)}(\zeta) d\zeta$$

$n - 1 < \alpha < n$ where $n \in \mathbb{N}^*$.

In the next section we will introduce the *Polynomial Least Square Method* (*PLSM*) which allows us to determine analytical approximate polynomial solutions for fractional ordinary differential equations and in the third section we will compare our approximate solutions with approximate solutions presented by *fractional Adams-Bashforth-Moulton method* (*FABMM*).

2 The Polynomial Least Squares Method

We denote by $\tilde{y}$ an approximate solution of equation (1). The error obtained by replacing the exact solution y with the approximation $\tilde{y}$ is given by the remainder:

$$\mathcal{R}(x, \tilde{y}(x)) = D^\alpha \tilde{y}(x) - f(x, \tilde{y}(x)). \tag{3}$$

For $\epsilon \in \mathbb{R}_+$, we will compute approximate polynomial solutions $\tilde{y}$ of the problem $(1, 2)$ on the interval $[0, b]$.

Definition 2.1. *We call an* ϵ-approximate polynomial solution *of the problem* $(1, 2)$ *an approximate polynomial solution* $\tilde{y}$ *satisfying the relations*

$$|\mathcal{R}(\tilde{y})| < \epsilon \tag{4}$$

$$\tilde{y}(0) = \nu_0. \tag{5}$$

We call a *weak* ϵ*-approximate polynomial solution* of the problem $(1, 2)$ an approximate polynomial solution $\tilde{y}$ satisfying the relation:

$$\int_0^b |\mathcal{R}(\tilde{y})| dx \leq \epsilon \tag{6}$$

together with the initial conditions (5).

Definition 2.2. *Let* $P_m(x) = c_0 + c_1x + c_2x^2 + \cdots + c_mx^m,\ c_i \in \mathbb{R},\ i = \overline{0, m}$ *be a sequence of polynomials satisfying the condition:*

$$P_m(0) = \nu_0.$$

We call the sequence of polynomials $P_m(x)$ *convergent to the solution of the problem* $(1, 2)$ *if* $\lim_{m\to\infty} D(P_m(x)) = 0$.

We observe that from the hypothesis of the initial problems $(1, 2)$ it follows that there exists a sequence of polynomials $P_m(x)$ which converges to the solution of the problem.

We will compute a weak ϵ - approximate polynomial solution, in the sense of the Definition 2.1, of the type:

$$\tilde{y}(x) = \sum_{k=0}^{m} d_k x^k \tag{7}$$

where $d_0, d_1, \cdots, d_m$ are constants which are calculated using the following steps:

- By substituting the approximate solution (7) in the equation (1) we obtain the expression:
$$\mathcal{R}(\tilde{y}) = D^{\alpha}\tilde{y}(x) - f(x, \tilde{y}(x)). \tag{8}$$
If we could find $d_0, d_1, \cdots, d_m$ such $\mathcal{R}(\tilde{y}) = 0,\ \tilde{y}(0) = \nu_0$, then by substituting $d_0, d_1, \cdots, d_m$ in (7) we obtain the solutions of equation (1).

- Then we attach to the problem (1,2) the following functional:
$$\mathcal{J}(d_1, d_2, d_3, \cdots, d_m) = \int_0^b \mathcal{R}^2(\tilde{y})dx \tag{9}$$
where d_0 is computed as functions of $d_1, d_2, d_3, \cdots, d_m$ using the initial condition (5).

- We compute the values $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ as the values which give the minimum of the functional $\mathcal{J}$, and the values of d_0 is function of $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ using the initial condition.

- With constants $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ previously determined we consider the polynomial:
$$M_m(x) = \sum_{k=0}^{m} d_k^0 x^k. \tag{10}$$

Theorem 2.1. *The sequence of polynomials $M_m(x)$ from (10) satisfies the property:*

$$\lim_{x\to\infty}\int_0^b \mathcal{R}^2(M_m(x))dx = 0. \tag{11}$$

Moreover, $\forall\epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ *it follows that* $M_m(x)$ *is a* weak ϵ-approximate polynomial solution *of the problem* $(1, 2)$.

Proof. Based on the way the polynomials $M_m(x)$ are computed and taking into account the relations (8)-(11), the following inequalities are satisfied:

$$0 \le \int_0^b \mathcal{R}^2(M_m(x))dx \le \int_0^b \mathcal{R}^2(P_m(x))dx,\ \forall m \in \mathbb{N},$$

where $P_m(x)$ is the sequence of polynomials introduced in Definition 2.2.

It follows that:

$$0 \le \lim_{x\to\infty}\int_0^b \mathcal{R}^2(M_m(x))dx \le \lim_{x\to\infty}\int_0^b \mathcal{R}^2(P_m(x))dx = 0.$$

We obtain:

$$\lim_{x\to\infty}\int_0^b \mathcal{R}^2(M_m(x))dx = 0.$$

From this limit we obtain that $\forall\epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ it follows that $M_m(x)$ is a *weak ϵ-approximate polynomial solution* of the problem $(1, 2)$. □

In order to find ϵ-approximate polynomial solutions of the problem (1,2) by using the Polynomial Least Squares Method we will first determine weak approximate polynomial solutions, $\tilde{y}$.

If $|\mathcal{R}(\tilde{y})| < \epsilon$ then $\tilde{y}$ is also an ϵ approximate polynomial solution of the problem.

3 Application

We consider the following linear fractional differential equation ([2]):

$$D^\alpha y(x) + y(x) - x^{\alpha+3} - \frac{\Gamma(4+\alpha)}{6} \cdot x^3 = 0 \tag{12}$$

$\alpha = 0,25$; $x \in [0, \frac{1}{30}]$ and the initial condition: $y(0) = 0$.

The exact solution of the problem is:

$$y(x) = x^{3+\alpha}.$$

A numerical solutions for this problem is presented by Baskonus at all in [2] using *fractional Adams-Bashfort-Moulton method* (*FABMM*).

Using (*PLSM*):

- We compute a solution of the type:

$$\tilde{y}(x) = d_0 + d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4$$

  with initial condition: $\tilde{y}(0) = 0$ we obtain: $d_0 = 0$.

- The approximate solution becomes:

$$\tilde{y}(x) = d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4.$$

- The corresponding remainder is:

$$\mathcal{R}(x) = \frac{4x^{3/4}\left(385d_1 + 8x\left(55d_2 + 60d_3x + 64d_4x^2\right)\right)}{1155\Gamma\left(\frac{3}{4}\right)} + \\ + d_1x + d_2x^2 + d_3x^3 + d_4x^4 - x^{13/4} - \frac{1}{6}x^3\Gamma\left(\frac{17}{4}\right). \tag{13}$$

  Next we compute:

$$\mathcal{J}(d_1, d_2, d_3, \cdots, d_m) = \int_0^{\frac{1}{30}} \mathcal{R}^2(\tilde{y})dx$$

  and minimize it obtaining the values:

$$d_1 = 3,53901 \cdot 10^{-6};\ d_2 = 0,00131029;\ d_3 = 0,387136,\ d_4 = 2,29079.$$

- The approximate analytical solution of the problem (12) using (*PLSM*) is:

$$\tilde{y}(x) = 3,53901 \cdot 10^{-6} \cdot x + 0,00131029 \cdot x^2 + 0,387136 \cdot x^3 + 2,29079 \cdot x^4.$$

Table 1 present the comparison between absolute errors coresponding to the numerical solution proposed by Baskonus in [2] using (*FABMM*) and aur solution (*PLSM*).

From the table, it is easy to see that using (*PLSM*) results are better than using (*FABMM*).

Additionally, (*PLSM*) obtains the analytical solution of the polynomial form of the problem, not only numerical solutions, thus demonstrating the usefulness and accuracy of the (*PLSM*).

Table 1: Numerical results

x	$Exactsolution$	$Error(FABMM)$	$Error(PLSM)$
0.0033333	2.82×10^{-3}	3.8343×10^{-9}	2.9598×10^{-9}
0.0066667	1.73×10^{-3}	2.1194×10^{-8}	7.4355×10^{-11}
0.0100000	3.31×10^{-4}	5.4419×10^{-8}	1.8279×10^{-9}
0.0133333	1.15×10^{-3}	1.0405×10^{-7}	1.1658×10^{-9}
0.0166667	1.75×10^{-3}	1.7047×10^{-7}	6.2667×10^{-10}
0.0200000	2.36×10^{-3}	2.5705×10^{-7}	1.8004×10^{-9}
0.0233333	1.49×10^{-3}	3.5512×10^{-7}	1.2389×10^{-9}
0.0266667	2.66×10^{-3}	4.7380×10^{-7}	7.2161×10^{-10}
0.0300000	4.88×10^{-3}	6.1050×10^{-7}	1.7042×10^{-9}
0.0333333	0	7.6535×10^{-7}	3.1652×10^{-9}

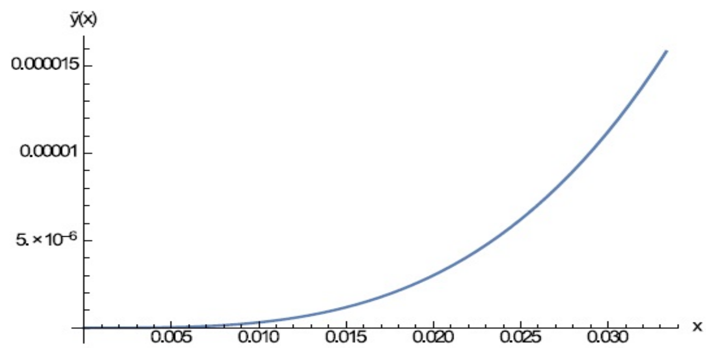

Figure 1 - The approximate analytical solution using (*PLSM*)

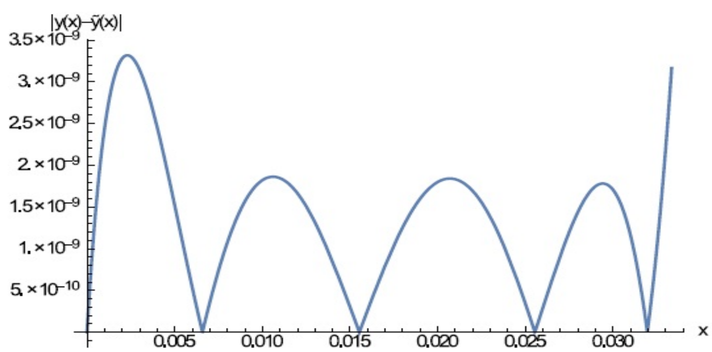

Figure 2 - The absolute errors corresponding to the approximations given by (*PLSM*)

4 Conclusions

The computations performed show that (*PLSM*) allows us to obtain approximations with an error relative to the exact or numerical solution smaller than the errors obtained using by *fractional Adams-Bashforth-Moulton method* (*FABMM*).

The application presented emphasize the high accuracy of the method by means of a comparison with previous results.

References

[1] C.Bota, B.Căruntu, *Analitic Approximate Solutions for a class of variable order fractional differential equations using the Polynomial Least Squares Method*, De Gruyter, Fractional Calculs and Applied Analysis, Vol. 20, no 4, 2017.

[2] H.M.Baskonus, H.Bulut, *On the numerical solutions of some fractional ordinary differential equations by fractional Adams-Bashforth-Moulton method* Open Math.2015; 13:547-556.

[3] I.Hashim, O. Abdulaziz, and S. Momani, *Homotopy analysis method for fractional IVPs*, Commun. Nonlinear Sci. Numer. Simul., Mar. 2009, Vol. 14, no. 3, 674-684 .

[4] S. Momani and K. Al-Khaled, *Numerical solutions for systems offractional differential equations by the decomposition method*, Appl. Math. Comput., Mar. 2005, Vol. 162, 1351-1365.

[5] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic, New York, 1974.

[6] I.Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[7] M.Rehman and R.A.Khan, *The Legendre wavelet method for solving fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., Nov. 2011, Vol. 16, no. 11, 4163-4173.

[8] L.Song and H.Zhang, *Solving the fractional BBM-Burgers equation using the homotopy analysis method*, Chaos Solitons Fractals, 40, 1616-1622, 2009.

[9] X.-J. Yang and D. Baleanu, *Fractal heat conduction problem solved by local fractional variation iteration method*, Thermal Science, 2013, vol. 17, no. 2, 625628.

[10] G.C. Wu, D. Baleanu and Z.G. Deng, *Variational iteration method as a kernel constructive technique*, Applied Mathematical Modelling, 39(15), 4378-4384, 2015.

Mădălina Sofia Paşca – Department of Mathematics,
Politehnica University of Timişoara,
Piaţa Victoriei 2, 300006 Timişoara, ROMANIA
E-mail: mspasca@yahoo.com

Marioara Lăpădat – Department of Mathematics,
Politehnica University of Timişoara,
Piaţa Victoriei 2, 300006, Timişoara, ROMANIA
E-mail: maria.lapadat@upt.ro

SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

LYAPUNOV FUNCTIONALS FOR SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES

Claudia - Luminiţa MIHIŢ

Abstract

The paper considers a notion of nonuniform splitting with growth rates for skew-evolution semiflows in Banach spaces. Characterizations for this concept are given through Lyapunov functionals with invariant and strongly invariant families of projections. [1]

Keywords and phrases: *Lyapunov functionals, skew-evolution semiflows, splitting*

1 Introduction. Preliminaries

The asymptotic property of (exponential) splitting was introduced by B. Aulbach and J. Kalkbrenner in [1] as a generalization of (exponential) dichotomy for difference equations. Regarding the qualitative results obtained for the dichotomy notion, we mention the contributions from [2], [4], [6] and the references therein.

Recent studies for more general concepts of splitting are made in [3] for noninvertible differential equations with impulse effect, respectively in [5] for skew-evolution semiflows.

The integral conditions represent an important tool to give criteria for asymptotic behaviours (see for instance [7], [8]). In this article, a result for nonuniform splitting with Lyapunov functionals is proved from the point of view of invariant families of projections, using an auxiliary integral characterization. Also, similar results are shown in the case of strongly invariant families of projections.

[1]MSC(2008): 34D05, 93D30

Let X be a real or complex Banach space and Θ a metric space. $\mathcal{B}(X)$ represents the Banach algebra of all bounded linear operators on X and the norms on X, respectively on $\mathcal{B}(X)$, will be denoted by $||\cdot||$. We consider the sets

$$\Delta=\{(t,s)\in\mathbb{R}_+^2 : t\geq s\},\quad T=\{(t,s,t_0)\in\mathbb{R}_+^3 : t\geq s\geq t_0\}$$

and $\Gamma=\Theta\times X$.

Definition 1.1. A continuous mapping $\varphi:\Delta\times\Theta\to\Theta$ is called *evolution semiflow* on Θ if the following relations hold:

(es_1) $\varphi(s,s,\theta)=\theta,$ for all $(s,\theta)\in\mathbb{R}_+\times\Theta$;
(es_2) $\varphi(t,s,\varphi(s,t_0,\theta))=\varphi(t,t_0,\theta),$ for all $(t,s,t_0,\theta)\in T\times\Theta$.

Definition 1.2. We say that $\Phi:\Delta\times\Theta\to\mathcal{B}(X)$ is *evolution cocycle* over the evolution semiflow φ if:

(ec_1) $\Phi(s,s,\theta)=I$ (the identity operator on X), for all $(s,\theta)\in\mathbb{R}_+\times\Theta$;
(ec_2) $\Phi(t,s,\varphi(s,t_0,\theta))\Phi(s,t_0,\theta)=\Phi(t,t_0,\theta),$ for all $(t,s,t_0,\theta)\in T\times\Theta$;
(ec_3) $(t,s,\theta)\mapsto\Phi(t,s,\theta)x$ is continuous for every $x\in X$.

Definition 1.3. If φ is evolution semiflow on Θ and Φ is evolution cocycle over the evolution semiflow φ, then the pair $C=(\varphi,\Phi)$ is called *skew-evolution semiflow* on Γ.

Definition 1.4. A continuous mapping $P:\mathbb{R}_+\times\Theta\to\mathcal{B}(X)$, which satisfies

$$P^2(t,\theta)=P(t,\theta),\quad\text{for all}\quad(t,\theta)\in\mathbb{R}_+\times\Theta,$$

is called *family of projections* on X.

If $P:\mathbb{R}_+\times\Theta\to\mathcal{B}(X)$ is a family of projections, then $Q:\mathbb{R}_+\times\Theta\to\mathcal{B}(X)$, $Q(t,\theta)=I-P(t,\theta)$ is the *complementary family of projections of* P.

Definition 1.5. A family of projections $P:\mathbb{R}_+\times\Theta\to\mathcal{B}(X)$ is called *invariant* for the skew-evolution semiflow $C=(\varphi,\Phi)$ if:

$$P(t,\varphi(t,s,\theta))\Phi(t,s,\theta)=\Phi(t,s,\theta)P(s,\theta),\quad\text{for all }(t,s,\theta)\in\Delta\times\Theta.$$

If in addition, for all $(t,s,\theta)\in\Delta\times\Theta$ the mapping $\Phi(t,s,\theta)$ is an isomorphism from *Range* $Q(s,\theta)$ to *Range* $Q(t,\varphi(t,s,\theta))$, then we say that P is *strongly invariant for* $C=(\varphi,\Phi)$.

Let $C = (\varphi, \Phi)$ be a skew-evolution semiflow, $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ an invariant family of projections for C and $h, k : \mathbb{R}_+ \to [1, +\infty)$ growth rates (i.e. nondecreasing functions with $\lim\limits_{t\to+\infty} h(t) = \lim\limits_{t\to+\infty} k(t) = +\infty$).

Definition 1.6. The pair (C, P) admits (h, k)-*splitting* if there exist $\alpha, \beta \in \mathbb{R}$, with $\alpha < \beta$ and a nondecreasing function $N : \mathbb{R}_+ \to [1, +\infty)$ such that:

(hs_1) $h(s)^\alpha ||\Phi(t, t_0, \theta)P(t_0, \theta)x|| \leq N(s)h(t)^\alpha ||\Phi(s, t_0, \theta)P(t_0, \theta)x||$;

(ks_1) $k(t)^\beta ||\Phi(s, t_0, \theta)Q(t_0, \theta)x|| \leq N(t)k(s)^\beta ||\Phi(t, t_0, \theta)Q(t_0, \theta)x||$,

for all $(t, s, t_0, \theta, x) \in T \times \Gamma$.

In particular, if $\alpha < 0 < \beta$, then we have the concept of (h, k)-*dichotomy*.

Definition 1.7. We say that (C, P) has (h, k)-*growth* if there exist $\omega > 0$ and a nondecreasing function $M : \mathbb{R}_+ \to [1, +\infty)$ with:

(hg_1) $h(s)^\omega ||\Phi(t, t_0, \theta)P(t_0, \theta)x|| \leq M(t_0)h(t)^\omega ||\Phi(s, t_0, \theta)P(t_0, \theta)x||$;

(kg_1) $k(s)^\omega ||\Phi(s, t_0, \theta)Q(t_0, \theta)x|| \leq M(t)k(t)^\omega ||\Phi(t, t_0, \theta)Q(t_0, \theta)x||$,

for all $(t, s, t_0, \theta, x) \in T \times \Gamma$.

Further, we recall some results obtained in [5].

Proposition 1.1. *If $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is a strongly invariant family of projections for $C = (\varphi, \Phi)$, then there exists an isomorphism $\Psi : \Delta \times \Theta \to \mathcal{B}(X)$ from Range $Q(t, \varphi(t, s, \theta))$ to Range $Q(s, \theta)$, such that:*

(Ψ^1) $\Phi(t, s, \theta)\Psi(t, s, \theta)Q(t, \varphi(t, s, \theta)) = Q(t, \varphi(t, s, \theta))$;

(Ψ^2) $\Psi(t, s, \theta)\Phi(t, s, \theta)Q(s, \theta) = Q(s, \theta)$;

(Ψ^3) $\Psi(t, s, \theta)Q(t, \varphi(t, s, \theta)) = Q(s, \theta)\Psi(t, s, \theta)Q(t, \varphi(t, s, \theta))$;

(Ψ^4) $\Psi(t, t_0, \theta)Q(t, \varphi(t, t_0, \theta)) = \Psi(s, t_0, \theta)\Psi(t, s, \varphi(s, t_0, \theta))Q(t, \varphi(t, t_0, \theta))$,

for all $(t, s, t_0, \theta) \in T \times \Theta$.

Proof. See [5], Proposition 2.9. □

We denote by $\mathcal{H}_1$ the set of all growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with

$$\int_0^{+\infty} h(s)^c ds < +\infty, \quad \text{for all } c < 0.$$

Also, $\mathcal{K}_1$ represents the set of all growth rates $k : \mathbb{R}_+ \to [1, +\infty)$, with the property that there exists a constant $K_1 \geq 1$ such that

$$\int_0^t k(s)^c ds \leq K_1 k(t)^c, \quad \text{for all } c > 0,\ t \geq 0.$$

By $\mathcal{H}$ we denote the set of all growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there exists $H > 1$ such that

$$1 \leq \frac{h(t+1)}{h(t)} < H, \quad \text{for all} \quad t \geq 0.$$

Theorem 1.1. *Let (C, P) be a pair with (h, k)-growth, where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C, P) admits (h, k)-splitting if and only if there exist $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$ and a nondecreasing mapping $D : \mathbb{R}_+ \to [1, +\infty)$ such that the following assertions hold:*

$$(Dhs_1) \qquad \int_s^{+\infty} \frac{||\Phi(\tau, t_0, \theta)P(t_0, \theta)x||}{h(\tau)^{d_1}} d\tau \leq \frac{D(s)}{h(s)^{d_1}} ||\Phi(s, t_0, \theta)P(t_0, \theta)x||,$$

for all $(s, t_0, \theta, x) \in \Delta \times \Gamma$;

$$(Dks_1) \qquad \int_{t_0}^{t} \frac{||\Phi(\tau, t_0, \theta)Q(t_0, \theta)x||}{k(\tau)^{d_2}} d\tau \leq \frac{D(t)}{k(t)^{d_2}} ||\Phi(t, t_0, \theta)Q(t_0, \theta)x||,$$

for all $(t, t_0, \theta, x) \in \Delta \times \Gamma$.

Proof. See [5], Theorem 3.2. □

2 The main results

Let $C = (\varphi, \Phi)$ be a skew-evolution semiflow, $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ an invariant family of projections for C and $h, k : \mathbb{R}_+ \to [1, +\infty)$ two growth rates.

Definition 2.1. We say that $L : T \times \Gamma \to \mathbb{R}_+$ is (h, k)-*Lyapunov functional* for the pair (C, P) if there exist two real constants $l_1 < l_2$ such that:

$$(hL_1) \quad \int_s^t \frac{||\Phi(\tau, t_0, \theta)P(t_0, \theta)x||}{h(\tau)^{l_1}} d\tau \leq \frac{L(s, s, t_0, \theta, P(t_0, \theta)x) - L(t, s, t_0, \theta, P(t_0, \theta)x)}{h(s)^{l_1}};$$

$$(kL_1) \quad \int_s^t \frac{||\Phi(\tau, t_0, \theta)Q(t_0, \theta)x||}{k(\tau)^{l_2}} d\tau \leq \frac{L(t, t, t_0, \theta, Q(t_0, \theta)x) - L(t, s, t_0, \theta, Q(t_0, \theta)x)}{k(t)^{l_2}},$$

for all $(t, s, t_0, \theta, x) \in T \times \Gamma$.

Theorem 2.1. *We consider (C, P) a pair with (h, k)-growth, where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C, P) admits (h, k)-splitting if and only if there exist $L : T \times \Gamma \to \mathbb{R}_+$ a (h, k)-Lyapunov functional for (C, P) and a nondecreasing function $\lambda : \mathbb{R}_+ \to [1, +\infty)$ with:*

(L_1) $L(s, s, t_0, \theta, P(t_0, \theta)x) \leq \lambda(s)||\Phi(s, t_0, \theta)P(t_0, \theta)x||$;
(L_2) $L(t, t, t_0, \theta, Q(t_0, \theta)x) \leq \lambda(t)||\Phi(t, t_0, \theta)Q(t_0, \theta)x||$,

for all $(t, s, t_0, \theta, x) \in T \times \Gamma$.

Proof. Necessity. Let $L : T \times \Gamma \to \mathbb{R}_+$ be defined by

$$L(t, s, t_0, \theta, x) = \int_t^{+\infty} \left(\frac{h(s)}{h(\tau)}\right)^{d_1} ||\Phi(\tau, t_0, \theta)P(t_0, \theta)x||d\tau +$$

$$+ \int_{t_0}^{s} \left(\frac{k(t)}{k(\tau)}\right)^{d_2} ||\Phi(\tau, t_0, \theta)Q(t_0, \theta)x||d\tau,$$

where $d_1 < d_2$ are given by Theorem 1.1.
It is immediate to see that the (hL_1) and (kL_1) from Definition 2.1 are satisfied. From Theorem 1.1, we deduce that (L_1) and (L_2) are verified.

Sufficiency. Using Definition 2.1, (hL_1), we have

$$\int_s^t \frac{||\Phi(\tau, t_0, \theta)P(t_0, \theta)x||}{h(\tau)^{l_1}} d\tau \leq \frac{L(s, s, t_0, \theta, P(t_0, \theta)x)}{h(s)^{l_1}} \leq$$

$$\leq \frac{\lambda(s)}{h(s)^{l_1}} ||\Phi(s, t_0, \theta)P(t_0, \theta)x||,$$

which implies

$$\int_s^{+\infty} \frac{||\Phi(\tau, t_0, \theta)P(t_0, \theta)x||}{h(\tau)^{l_1}} d\tau \leq \frac{\lambda(s)}{h(s)^{l_1}} ||\Phi(s, t_0, \theta)P(t_0, \theta)x||, \tag{1}$$

for all $(s, t_0, \theta, x) \in \Delta \times \Gamma$.
Similarly, from (kL_1), for $t_0 = s$ it follows

$$\int_{t_0}^{t} \frac{||\Phi(\tau,t_0,\theta)Q(t_0,\theta)x||}{k(\tau)^{l_2}}d\tau \leq \frac{L(t,t,t_0,\theta,Q(t_0,\theta)x)}{k(t)^{l_2}}$$

and then

$$\int_{t_0}^{t} \frac{||\Phi(\tau,t_0,\theta)Q(t_0,\theta)x||}{k(\tau)^{l_2}}d\tau \leq \frac{\lambda(t)||\Phi(t,t_0,\theta)Q(t_0,\theta)x||}{k(t)^{l_2}}, \tag{2}$$

for all $(t,t_0,\theta,x) \in \Delta \times \Gamma$.
From (1), (2) and Theorem 1.1 we obtain that (C,P) has (h,k)-splitting. $\square$

In what follows, $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ represents a strongly invariant family of projections for C and $\Psi : \Delta \times \Theta \to \mathcal{B}(X)$ is given by Proposition 1.1.

Proposition 2.1. *The mapping $L : T \times \Gamma \to \mathbb{R}_+$ is (h,k)-Lyapunov functional for the pair (C,P) if and only if there exist $l_1, l_2 \in \mathbb{R}$, $l_1 < l_2$ with the properties:*

$$(hL_1) \quad \int_{s}^{t} \frac{||\Phi(\tau,t_0,\theta)P(t_0,\theta)x||}{h(\tau)^{l_1}}d\tau \leq \frac{L(s,s,t_0,\theta,P(t_0,\theta)x) - L(t,s,t_0,\theta,P(t_0,\theta)x)}{h(s)^{l_1}};$$

$$(kL_1') \quad \int_{s}^{t} \frac{||\Psi(t,\tau,\varphi(\tau,t_0,\theta))Q(t,\varphi(t,t_0,\theta))x||}{k(\tau)^{l_2}}d\tau \leq$$

$$\leq \frac{L(t,t,t_0,\theta,\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))x) - L(t,s,t_0,\theta,\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))x)}{k(t)^{l_2}},$$

for all $(t,s,t_0,\theta,x) \in T \times \Gamma$.

Proof. It is sufficient to justify the equivalence $(kL_1) \Leftrightarrow (kL_1')$ and we use the relations from Proposition 1.1.
If (kL_1) holds, then for all $(t,s,t_0,\theta,x) \in T \times \Gamma$ we have:

$$\int_{s}^{t} \frac{||\Psi(t,\tau,\varphi(\tau,t_0,\theta))Q(t,\varphi(t,t_0,\theta))x||}{k(\tau)^{l_2}}d\tau =$$

$$= \int_{s}^{t} \frac{||Q(\tau,\varphi(\tau,t_0,\theta))\Psi(t,\tau,\varphi(\tau,t_0,\theta))Q(t,\varphi(t,t_0,\theta))x||}{k(\tau)^{l_2}}d\tau =$$

$$= \int_s^t \frac{||\Phi(\tau,t_0,\theta)\Psi(\tau,t_0,\theta)Q(\tau,\varphi(\tau,t_0,\theta))\Psi(t,\tau,\varphi(\tau,t_0,\theta))Q(t,\varphi(t,t_0,\theta))x||}{k(\tau)^{l_2}}d\tau =$$

$$= \int_s^t \frac{||\Phi(\tau,t_0,\theta)\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))x||}{k(\tau)^{l_2}}d\tau =$$

$$= \int_s^t \frac{||\Phi(\tau,t_0,\theta)Q(t_0,\theta)\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))x||}{k(\tau)^{l_2}}d\tau \leq$$

$$\leq \frac{L(t,t,t_0,\theta,Q(t_0,\theta)\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))x)}{k(t)^{l_2}} -$$

$$- \frac{L(t,s,t_0,\theta,Q(t_0,\theta)\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))x)}{k(t)^{l_2}} =$$

$$= \frac{L(t,t,t_0,\theta,\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))x) - L(t,s,t_0,\theta,\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))x)}{k(t)^{l_2}}.$$

Conversely, if (kL_1') is satisfied, then

$$\int_s^t \frac{||\Phi(\tau,t_0,\theta)Q(t_0,\theta)x||}{k(\tau)^{l_2}}d\tau = \int_s^t \frac{||Q(\tau,\varphi(\tau,t_0,\theta))\Phi(\tau,t_0,\theta)Q(t_0,\theta)x||}{k(\tau)^{l_2}}d\tau =$$

$$= \int_s^t \frac{||\Psi(t,\tau,\varphi(\tau,t_0,\theta))\Phi(t,\tau,\varphi(\tau,t_0,\theta))Q(\tau,\varphi(\tau,t_0,\theta))\Phi(\tau,t_0,\theta)Q(t_0,\theta)x||}{k(\tau)^{l_2}}d\tau =$$

$$= \int_s^t \frac{||\Psi(t,\tau,\varphi(\tau,t_0,\theta))Q(t,\varphi(t,t_0,\theta))\Phi(t,t_0,\theta)Q(t_0,\theta)x||}{k(\tau)^{l_2}}d\tau \leq$$

$$\leq \frac{L(t,t,t_0,\theta,\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))\Phi(t,t_0,\theta)Q(t_0,\theta)x)}{k(t)^{l_2}} -$$

$$- \frac{L(t,s,t_0,\theta,\Psi(t,t_0,\theta)Q(t,\varphi(t,t_0,\theta))\Phi(t,t_0,\theta)Q(t_0,\theta)x)}{k(t)^{l_2}} =$$

$$= \frac{L(t,t,t_0,\theta,Q(t_0,\theta)x) - L(t,s,t_0,\theta,Q(t_0,\theta)x)}{k(t)^{l_2}},$$

for all $(t,s,t_0,\theta,x) \in T \times \Gamma$. $\square$

Theorem 2.2. *Let (C, P) be a pair with (h, k)-growth, $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C, P) has (h, k)-splitting if and only if there exist $L : T \times \Gamma \to \mathbb{R}_+$ a (h, k)-Lyapunov functional for (C, P) and a nondecreasing mapping $\lambda : \mathbb{R}_+ \to [1, +\infty)$ such that:*

(L_1) $L(s, s, t_0, \theta, P(t_0, \theta)x) \leq \lambda(s)||\Phi(s, t_0, \theta)P(t_0, \theta)x||$;

(L_2') $L(t, t, t_0, \theta, \Psi(t, t_0, \theta)Q(t, \varphi(t, t_0, \theta))x) \leq \lambda(t)||Q(t, \varphi(t, t_0, \theta))x||$,

for all $(t, s, t_0, \theta, x) \in T \times \Gamma$.

Proof. We show the equivalence between the conditions (L_2) and (L_2'), using Proposition 1.1.
If (L_2) is verified, then we deduce:

$$L(t, t, t_0, \theta, \Psi(t, t_0, \theta)Q(t, \varphi(t, t_0, \theta))x) =$$

$$= L(t, t, t_0, \theta, Q(t_0, \theta)\Psi(t, t_0, \theta)Q(t, \varphi(t, t_0, \theta))x) \leq$$

$$\leq \lambda(t)||\Phi(t, t_0, \theta)Q(t_0, \theta)\Psi(t, t_0, \theta)Q(t, \varphi(t, t_0, \theta))x|| =$$

$$= \lambda(t)||Q(t, \varphi(t, t_0, \theta))x||,$$

for all $(t, t_0, \theta, x) \in \Delta \times \Gamma$.
In a similar manner, if (L_2') holds, then we obtain:

$$L(t, t, t_0, \theta, Q(t_0, \theta)x) = L(t, t, t_0, \theta, \Psi(t, t_0, \theta)\Phi(t, t_0, \theta)Q(t_0, \theta)x) =$$

$$= L(t, t, t_0, \theta, \Psi(t, t_0, \theta)Q(t, \varphi(t, t_0, \theta))\Phi(t, t_0, \theta)Q(t_0, \theta)x) \leq$$

$$\leq \lambda(t)||Q(t, \varphi(t, t_0, \theta))\Phi(t, t_0, \theta)Q(t_0, \theta)x|| = \lambda(t)||\Phi(t, t_0, \theta)Q(t_0, \theta)x||,$$

for all $(t, s, t_0, \theta, x) \in T \times \Gamma$. □

Acknowledgements. The author would like to express special thanks to Professor Emeritus Mihail Megan for the suggestions and support offered in preparing this article.

References

[1] B. Aulbach, J. Kalkbrenner, Exponential forward splitting for noninvertible difference equations, *Comput. Math. Appl.* **42** (2001), 743-754.

[2] A. J. G. Bento, N. Lupa, M. Megan, C. Silva, Integral conditions for nonuniform μ- dichotomy, *Discrete Contin. Dyn. Syst. Ser. B*, **22 (8)**, (2017), 3063-3077.

[3] J. L. Fenner, On forward (h, k) splitting for a class of noninvertible differential equations with impulse effect, *Nonlinear Anal.* **67** (2007), 1513-1521.

[4] M. Megan, C. Stoica, Concepts of dichotomy for skew-evolution semiflows in Banach spaces, *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **2(2)**, (2010), 125-140.

[5] C. L. Mihiţ, M. Megan, Integral characterizations for the (h, k)-splitting of skew-evolution semiflows, *Stud. Univ. Babeş-Bolyai Math.* **62(3)** (2017), 353-365.

[6] R. J. Sacker, G. R. Sell, Dichotomies for linear evolutionary equations in Banach spaces, *J. Differential Equations* **113** (1994), 17-67.

[7] A. L. Sasu, B. Sasu, Integral equations in the study of the asymptotic behavior of skew-product flows, *Asymptot. Anal.* **68** (2010), 135-153.

[8] C. Stoica, M. Megan, On (h, k)-trichotomy for skew-evolution semiflows in Banach spaces, *Stud. Univ. Babeş-Bolyai Math.* **56** (2011), 147-156.

Claudia Luminiţa Mihiţ,
Department of Mathematics,
West University of Timişoara,
V. Pârvan Blv. 4, 300223, Timişoara, ROMANIA
E-mail: mihit.claudia@yahoo.com

SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

A NEW APPROACH OF REFRACTION FOR 3D ELECTRIC FIELD IN NONLINEAR DIELECTRICS WITH PERMANENT POLARIZATION AND RANDOM ANISOTROPY
Part III. Applications of the new refraction theorems for particular cases

Ioan BERE

Abstract

Using a new permittivity - defined by author (in Part I) for dielectrics with permanent polarization we will demonstrate new theorems of refraction (in Part II), more general, for three-dimensional (3D) electric field lines at the separation surface of two nonlinear and anisotropic materials with permanent polarization, which have random polarization main directions. Then (in Part three), some applications of the new refraction theorems are presented, for particular cases. [1]

1 Applications of the new refraction theorems for particular cases.

1.1 Fields 3D in nonlinear and isotropic dielectrics, with permanent polarization.

For isotropic dielectrics, the components of calculation permittivity in the two materials are:

$$\varepsilon_{p1x} = \varepsilon_{p1y} = \varepsilon_{p1z} = \varepsilon_{p1}\,, \tag{1.1}$$

$$\varepsilon_{p2x} = \varepsilon_{p2y} = \varepsilon_{p2z} = \varepsilon_{p2}\,. \tag{1.2}$$

If we take into account equations (1.1) and (1.2), the theorem (31) from [10] for refraction of electric field strength lines becomes

$$\begin{gathered}\varepsilon_{p1}(\,E_{1xn} + E_{1yn} + E_{1zn}) - \,\varepsilon_{p2}(\,E_{2xn} + E_{2yn} + E_{2zn}) + \\ +(P_{p1xn} + P_{p1yn} + P_{p1zn}) - (P_{p2xn} + P_{p2yn} + P_{p2zn}) = 0\,.\end{gathered} \tag{1.3}$$

[1]Keywords and phrases: *a new permittivity, permanent polarization, random anisotropy, 3D refraction theorems*

Considering the signification from theorem (31) from [10], the expression (3) may be written shortly in this way:

$$\varepsilon_{p1}\, E_{1n} = \varepsilon_{p2}\, E_{2n} - (P_{p1n} - P_{p2n})\,. \tag{1.4}$$

Alike, taking into account equations (1.1) and (1.2), the theorem (31) from [10], for refraction of calculation electric flux density lines, becomes:

$$\frac{1}{\varepsilon_{p1}}(D_{p1xt} + D_{p1yt} + D_{p1zt}) - \frac{1}{\varepsilon_{p2}}(D_{p2xt} + D_{p2yt} + D_{p2zt}) = 0. \tag{1.5}$$

Considering the signification from theorem (36) from [10], the expression (1.5) can be written in a more concise form:

$$\frac{D_{p1t}}{\varepsilon_{p1}} = \frac{D_{p2t}}{\varepsilon_{p2}}\,. \tag{1.6}$$

Equations (1.4) and (1.6) represent the theorems of refraction for $\overline{E}$ and $\overline{D}_p$ in 3D fields, for nonlinear and isotropic dielectrics, with permanent polarization. We can remark that, *the theorem (6) of refraction in dielectrics with permanent polarization (*for the tangent components of $\overline{D}_p$*) has a similar form* (but another content) *with the classical theorem of refraction in materials without permanent polarization.*

Also, *the theorem (6) has a more simple form than classical treatment for refraction of electric flux density lines*$\overline{D}$, in nonlinear and isotropic dielectrics with permanent polarization (see [6], eq. (31)). *This simple form occurs as a result of the introduction of new quantities* $\overline{D}_p$*and* $\overline{\overline{\varepsilon}}_p$.

1.2 Fields 3D in nonlinear and isotropic dielectrics without permanent polarization

For dielectrics without permanent polarization ($\overline{P}_p = 0$), from eq. (1.5) from [9] we obtain $\overline{D}_p = \overline{D}$. Also, from eq. (1.8) from [9], for isotropic media we can write $\varepsilon_p = D_p/E = D/E$. So $\varepsilon_p = \varepsilon$, which means that (if the dielectric is without permanent polarization)the calculation permittivity is identical with the classical permittivity.

Particularizing eq.(1.4) and (1.6) for this case and taking into account the previous observation, we obtain

$$\frac{\varepsilon_{p1}}{\varepsilon_{p2}} = \frac{D_{p1t}}{D_{p2t}} = \frac{E_{2n}}{E_{1n}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{D_{1t}}{D_{2t}}. \tag{1.7}$$

The dielectrics are isotropic and therefore $\overline{D}_p$ and $\overline{E}$ have the same spectrum. Since $\overline{D}_p$ and $\overline{D}$ are identical, it follows that $\overline{D}$ and $\overline{E}$ have the same spectrum.

1.3 Fields 2D in nonlinear and anisotropic/isotropic dielectrics with permanent polarization

In the case of nonlinear and anisotropic or isotropic dielectrics, for two-dimensional (2D) field, vectors $\overline{D}_p$, $\overline{E}$ and $\overline{P}_p$ have not the components after z axis, but only after x and y axes. Eq. (1.4) and (1.6) are valid in this case, but z components missing from the detailed eq. (1.3) and (1.5).

For *2D fields, in anisotropic dielectrics by orthogonal directions*, if we represent $\overline{D}_{p\lambda}$ and $\overline{E}_\lambda$ vectors and their normal and tangential components to the surfaceS_{12}, we obtain the representations of Figure 1 (for $\overline{D}_p$) and Figure 2 (for $\overline{E}$). These are analogous to classical representation, but $\overline{D}_p$ in place to $\overline{D}$ (see [8], Figure 2 and Figure 3).

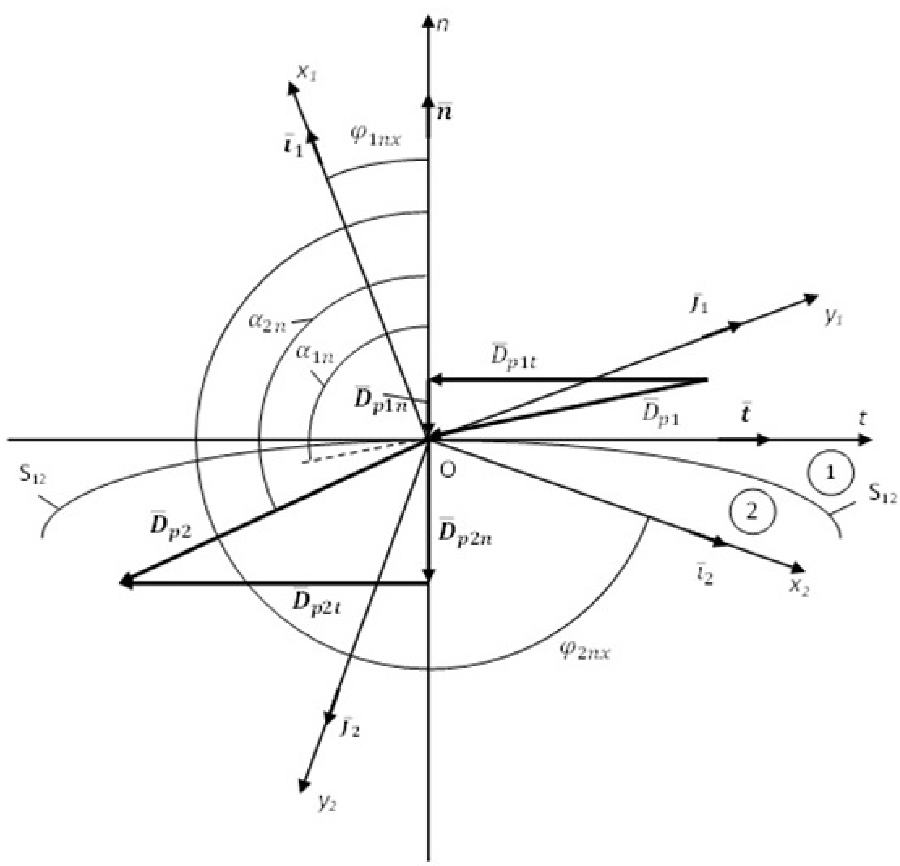

Figure 1– Refraction of $\overline{D}_p$

For *isotropic dielectrics* $\overline{D}_p$ *and* $\overline{E}$ *have the same spectrum* and would obtain similar representations to those in Figure 3 and Figure 4, but with
$\alpha_\lambda = \beta_\lambda\,, \quad (\lambda = 1,\ 2)$.

We remark that *the classical quantities* $\overline{D}$ *and* $\overline{E}$ *have not the same spectrum* (even if the dielectrics are isotropic) *because it is permanent polarization.*

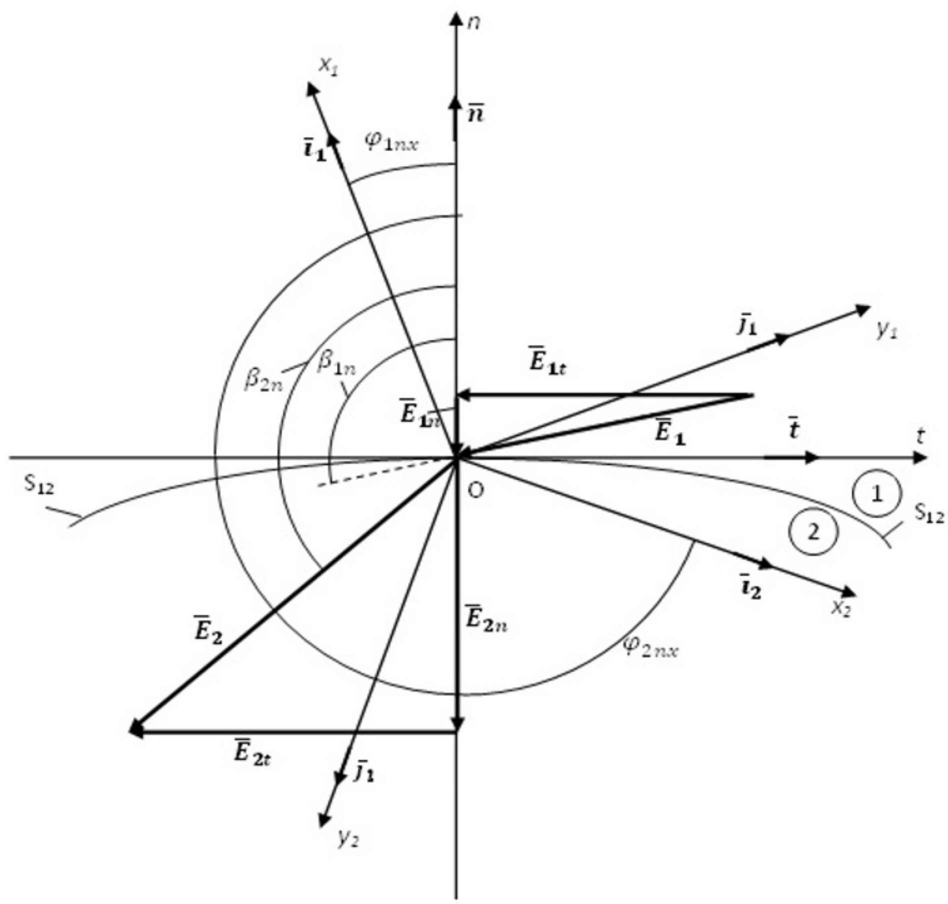

Figure 2– Refraction of $\overline{E}$

1.4 Fields 2D in nonlinear and isotropic dielectrics without permanent polarization

In this case $\overline{D}_{p\lambda} = \overline{D}_\lambda\,, \quad \varepsilon_{p\lambda} = \varepsilon_\lambda\,, \quad \alpha_{\lambda n} = \beta_{\lambda n}\,, \quad \alpha_{\lambda t} = \beta_{\lambda t}$ and $\alpha_{\lambda n} + \alpha_{\lambda t} = 90^0 (\lambda = 1,\ 2\,)$. Taking into account and the classical representation for 2D field refraction in isotropic dielectrics without permanent polarization [1, 2, 3], with these specifications, eq. (1.7) can be completed, being found and classical expressions:

$$\frac{\varepsilon_{p1}}{\varepsilon_{p2}} = \frac{D_{p1t}}{D_{p2t}} = \frac{E_{2n}}{E_{1n}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{D_{1t}}{D_{2t}} = \frac{tg\ \alpha_{1n}}{tg\ \alpha_{2n}} = \frac{tg\ \beta_{1n}}{tg\ \beta_{2n}}. \tag{1.8}$$

2 Other specifications

From the general expressions of refraction theorems for lines of vectors $\overline{E}$ and $\overline{D}_p$, or from particular forms already mentioned, can be obtained also other particular forms. Such cases are possible when the permanent polarization vectors $\overline{P}_p$ have particular orientations, when one of the dielectrics has permanent polarization and the other one does not (for example: dielectric with permanent polarization – air, dielectric with permanent polarization – ordinary dielectric without permanent polarization and so on), or when the polarization main directions are particular orientations(for example, rectangular directions) etc.

If known the electric hysteresis cycle for the isotropic dielectric, we should determine the nonlinear function$D_p(E)$. Then, it can be determined the diagram of nonlinear function $\varepsilon_{rp}(E)$ (or $\varepsilon_p(E)$), following the procedure used by the author for the permeability of permanent magnets (see [5], [7]). For an anisotropic dielectric with permanent polarization, the electric hysteresis cycles must be known by the all main directions of polarization. In this case, it can be determined (following similar procedures) the diagrams of nonlinear functions $D_{p\nu}(E)$ and $\varepsilon_{rp\nu}(E)$, where $\nu = x\,,\ y\,,\ z$.

It notes that similar theorems were demonstrated by the same author (see [5], [7]) for the *magnetic field lines refraction in materials with permanent magnetization* (i.e. permanent magnets). If compare the two situations, is *remarkable analogy* between the equations for electric field refraction in dielectrics with permanent polarization, respectively the equations for magnetic field refraction in permanent magnets.

3 Conclusions

Referring to the entire work (Part one, Part two, Part three) it highlights the following conclusions:

The introduction of calculation flux density $\overline{D}_p$ and of new relative permittivity $\overline{\overline{\varepsilon}}_{rp}$ for anisotropic dielectrics, nonlinear and with permanent polarization (in Part one, namely [9]) is a useful operation, because *the solution of field problem can be obtained in an advantageous way.*

In applications that refer to new forms of the theorems for 3D electric field lines refractions in nonlinear dielectrics, with permanent polarization and random anisotropy (in Part two, namely [10]), *the equations obtained are more concise, so simpler*. It is possible to make and *useful analogies* with the simpler case of the materials without permanent polarization.

As applications of new defined quantities, for anisotropic dielectrics with random polarization main directions and with permanent polarization, the author has demonstrated (in Part two, namely [10]) *new refraction theorems for 3D electric field* (eq. (31) from [10] for electric field strength $\overline{E}$, respectively eq. (36) from [10] for calculation electric flux density $\overline{D}_p$).

Starting from these general forms of the theorems, some particular forms have been deduced, in this paper (Part three). These can be useful in solving the electric field problems for nonlinear, anisotropic systems and with permanent polarization.

References

[1] C. Şora, Bazele electrotehnicii, *E.D.P.*, Bucureşti, pp. 63-67; 153-157, (1982).

[2] C. I. Mocanu, Teoria cmpului electromagnetic, *E.D.P.*, Bucureşti, pp. 183-187, (1981).

[3] M. Preda et all, Bazele electrotehnicii, *E.D.P.*, Bucureşti, pp. 82-85; 142-143, (1980).

[4] K. H. Bachmann et all, Kleine Enzyklopadie der Mathematik, *Editura Tehnică*, Bucureşti, pp 665–672, (1980).

[5] I. Bere, A new permeability for permanent magnets and another theorem of refraction in isotropic materials with permanent magnetization, *Periodica Polytechnica – Electrical Engineering*, Budapest, vol. 45/1, pp. 3-12, (2001).

[6] I. Bere, The theorems of refraction for 3D electric field in anisotropic dielectrics with permanent polarization, having random polarization main directions, *Proceedings of the 7th International Power Systems Conference,* Timişoara, pp. 61-68, (2007).

[7] I. Bere, A new approach of refraction for 3D field in anisotropic permanent magnets with random magnetization main directions, *Differential Equations and Control Processes,* Electronic Journal, Saint–Petersburg, no. 1, pp. 1–9, (2010).

[8] I. Bere, The permittivity for anisotropic dielectrics with permanent polarization, *Progress In Electromagnetics Research M,* Electronic Journal, Cambridge MA, USA, Vol. 23, pp. 263-277, (2012).

[9] I. Bere, A new approach of refraction for 3D electric field in nonlinear dielectrics with permanent polarization and random anisotropy. Part one: A new permittivity for dielectric with permanent polarization, *Scientific Bulletin of the Politehnica University of Timişoara, Transactions on Mathematics and Physics,* Vol. 61(75), Issue 2, pp. 33-38, (2016).

[10] I. Bere, A new approach of refraction for 3D electric field in nonlinear dielectrics with permanent polarization and random anisotropy. Part two: A new permittivity for dielectric with permanent polarization, *Scientific Bulletin of the Politehnica University of Timişoara, Transactions on Mathematics and Physics,* Vol. 62(76), Issue 2, pp. 22-30, (2017).

Ioan Bere
Department of Physical Foundation of Engineering
Politehnica University of Timisoara
V. Parvan Bv. No. 2, 300223 - Timisoara, ROMANIA
E-mail: ioan.bere@et.upt.ro

INSTRUCTIONS FOR THE AUTHORS

The "Buletinul Ştiinţific al Universităţii Politehnica Timişoara" is a direct successor of the "Buletin scientifique de l'École Polytechnique de Timişoara" which was started in 1925. Between 1982 – 1989 it was published as the "Lucrările Seminarului de Matematică şi Fizică ale Institutului Politehnic "Traian Vuia" din Timişoara".

Publication program: one volume per year in two issues, each series.

The "Mathematics-Physics" series of the "Buletinul Ştiinţific al Universităţii Politehnica Timişoara" publishes original papers in all areas of the pure and applied mathematics and physics.

1. The manuscript should be sent to the Editor, written in English (or French, or German, or Russian). The manuscript should be prepared in LATEX.
2. Maximum length of the paper is 8 pages, preferably in an even number of pages, in Times New Roman (12 pt).
3. For the first page the authors must bear in view:
   - an Abstract single spaced (at most 150 words) will be placed before the beginning of the text as: ABSTRACT. The problem of ...
   - footnote with MSC (Mathematics Subjects Classification) or PACS (Physics Abstracts Classification System) Subject Classification Codes (10 pt)
4. Graphs. illustrations and tables should placed into the manuscript (send it as .eps or .pdf file), with the corresponding consecutively number and explanations under them.
5. The complete author's (authors') address(es) will be placed after References.
6. A Copyright Transfer Agreement is required together with the paper. By submitting a paper to this journal, authors certify that the manuscript has not been submitted to, nor is it under consideration for publication by another journal, conference proceedings, or similar publications.
7. There is no page charge and after registration a paper is sent to two independent referees. After acceptance and publication the author will receive 5 reprints free of charge. If a larger amount is required (fee of 2 USD per copy per article) this should be communicated to the Editor.
8. Manuscripts should be sent to:

For Mathematics
Dr. Liviu CĂDARIU
POLITEHNICA UNIVERSITY TIMISOARA
Department of Mathematics
Victoriei Square, No. 2
300006 – TIMIȘOARA, ROMANIA
liviu.cadariu-brailoiu@mat.upt.ro

For Physics
Dr. Dušan POPOV
POLITEHNICA UNIVERSITY TIMISOARA
Department of Physical Fundamentals of Engineering
B-dul. V. Parvan, No.2
300223 – TIMIȘOARA, ROMANIA
dusan.popov@et.upt.ro